SCORZA QUARTICS OF TRIGONAL SPIN CURVES AND THEIR VARIETIES OF POWER SUMS

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ABSTRACT. Our fundamental result is the construction of new subvarieties in the varieties of power sums for the Scorza quartic of any general pairs of trigonal curves and non-effective theta characteristics. This is a generalization of Mukai's description of smooth prime Fano threefolds of genus twelve as the varieties of power sums for plane quartics. Among other applications, we give an affirmative answer to the conjecture of Dolgachev and Kanev on the existence of the Scorza quartic for any general pairs of curves and non-effective theta characteristics.

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Date: 1.11, 2007.
1991 *Mathematics Subject Classification.* Primary 14J45; Secondary 14N05, 14H42.
Key words and phrases. Waring problem, Variety of power sums, theta characteristic, Scorza quartic, two ray game.

1. Introduction

1.1. **Varieties of power sums and the Waring problem.**

Throughout the paper, we work over $\mathbb{C}$, the complex number field.

The problem of representing a homogeneous form as a sum of powers of linear forms has been studied since the last decades of the 19^{th} century. This is called the *Waring problem* for a homogeneous form. We are interested in the study of the global structure of a suitable compactification of the variety parameterizing all such representations of a homogeneous form. To give a precise definition of such a compactification consider a $(v+1)$-dimensional vector space V. Let $F \in S^m\check{V}$ be a homogeneous forms of degree m on V, where $\check{V}$ is the dual vector space of V. Let $\mathbb{P}_*\check{V}$ be the projective space parameterizing one-dimensional vector subspaces in $\check{V}$, which is sometime denoted by $\check{\mathbb{P}}^v$.

Definition 1.1.1. The *varieties of power sums* of F is the following set with reduced structure:

$$\mathrm{VSP}\,(F,n) := \overline{\{([H_1],\dots,[H_n]) \mid H_1^m + \cdots + H_n^m = F\}} \subset \mathrm{Hilb}^n(\mathbb{P}_*\check{V}).$$

We call the *Waring rank of* F the minimum of n such that $\mathrm{VSP}\,(F,n) \neq \emptyset$.

There are other compactifications, for example, the one in the n-th symmetric product of $\mathbb{P}_*\check{V}$, but for our treatment we need the one in the Hilbert scheme.

As far as we know, the first global descriptions of positive dimensional varieties of power sums for some homogeneous forms were given by S. Mukai.

The most intensively studied cases of varieties of power sums, including Mukai's case, are where F is a general $(v+1)$-nary homogeneous form of degree m for some $m, v \in \mathbb{N}$, and n is the Waring rank of F, which we denote by $n(m,v)$.

By a standard parameter count, we can easily compute the expected dimension of $\mathrm{VSP}\,(F,n)$ for a general homogeneous form F. Since the dimension of the vector space of $(v+1)$-nary homogeneous forms of degree m is $\binom{m+v}{m}$, the expected dimension is

$$\mathrm{expdim}\,\mathrm{VSP}\,(F,n) := (v+1)n - \binom{m+v}{m}.$$

Thus
it is expected that

$$n(m,v) = \lceil \frac{1}{v+1}\binom{m+v}{m} \rceil.$$

It is known, however, that there are exceptions to $n(m,v)$ by the following result of J. Alexander and A. Hirschowitz [AH95]:

m	v	$n(m,v)$
2	arbitrary	$v+1$
3	4	8
4	2	6
4	3	10
4	4	15

Here is the table of the known descriptions of $\mathrm{VSP}\,(F, n(m,v))$.

m	v	$n(m,v)$	VSP $(F, n(m,v))$	Ref.
$2a-1$	1	a	1 point	Sylvester
2	2	3	quintic del Pezzo threefold	Mukai [Muk92]
3	2	4	$\mathbb{P}^2$	Dolgachev and Kanev [DK93]
4	2	6	prime Fano threefold of genus twelve	[Muk92]
5	2	7	1 point	Hilbert, Richmond, Palatini
6	2	10	polarized $K3$ surface of genus 20	[Muk92]
7	2	12	5 points	Dixon and Stuart
8	2	15	16 points	[Muk92]
2	3	4	$G(2,5)$	Ranestad and Schreier [RS00]
3	3	5	1 point	Sylvester's Pentahedral Theorem
3	4	8	W	[RS00]
3	5	10	S	Iliev and Ranestad [IR01b]

In the table,

- W is a fivefold and is the variety of lines in the fivefold linear complete intersection $\mathbb{P}^{10} \cap \mathrm{OG}(5,10) \subset \mathbb{P}^{15}$ of the ten-dimensional orthogonal Grassmaniann $\mathrm{OG}(5,10)$,
- S is a smooth symplectic fourfold obtained as a deformation of the Hilbert square of a polarized $K3$ surface of genus eight, and
- see the introduction of [RS00] or [Dol04] for the references of the results in the 19^{th} and early 20^{th} centuries.

As we can see in the table, the study before Mukai's one were devoted only to the cases where $\dim \mathrm{VSP}(F, n(m,v)) = 0$ and mostly the cases where F has a unique representation. Recently, using the technique of birational geometry, M. Mella proved in [Mel06] that, if $m > v > 1$, then the uniqueness holds only in the case where $(m,v) = (5,2)$.

In [IR01a], Iliev and Ranestad treat some special $(v+1)$-nary cubics F and prove that, if $v \geq 8$, then the Waring rank of F is less than that of a general cubic.

In [IK99] Iliev and Kanev study varieties of power sums more systematically.

1.2. Mukai's contribution.

Let V_{22} be a smooth prime Fano threefold of genus twelve, namely, a smooth projective threefold such that $-K_{V_{22}}$ is ample, the class of $-K_{V_{22}}$ generates $\operatorname{Pic} V_{22}$, and the genus $g(V_{22}) := \frac{(-K_{V_{22}})^3}{2} + 1$ is equal to twelve. V_{22} can be embedded into $\mathbb{P}^{13}$ by the linear system $|-K_{V_{22}}|$. Mukai discovered the following remarkable result [Muk92, §6, Theorem 11] (see also [DK93], [Sch01], and [Dol04, Theorem 3.12] for some details):

Theorem 1.2.1. *For a general ternary quartic form F_4, $\mathrm{VPS}(F_4, 6) \subset \mathrm{Hilb}^6 \check{\mathbb{P}}^2$ is a smooth prime Fano threefold of genus twelve, where we use the dual notation for later convenience. Moreover every general V_{22} is of this form.*

To characterize a general V_{22} he studied the Hilbert scheme of lines on a general $V_{22} \subset \mathbb{P}^{13}$ showing that it is isomorphic to a smooth plane quartic curve $\mathcal{H}_1 \subset \mathbb{P}^2$. He thought how to recover V_{22} by $\mathcal{H}_1$. For this, one more data was necessary. Using the incidence relation on $\mathcal{H}_1 \times \mathcal{H}_1$ defined by intersections of lines on V_{22}, he found a non-effective theta characteristic θ on $\mathcal{H}_1$. As explained in [DK93, §6,7], there is a beautiful result of G. Scorza which asserts that, associated to the pair $(\mathcal{H}_1, \theta)$,

there exists another plane quartic curve $\{F_4 = 0\}$ in the same ambient plane as $\mathcal{H}_1$. (By saluting Scorza, $\{F_4 = 0\}$ is called the *Scorza quartic*.) Then, finally, Mukai proved that V_{22} is recovered as $\mathrm{VSP}\,(F_4, 6)$. Mukai observed that conics on V_{22} are parameterized by the plane $\mathcal{H}_2$ and $\mathcal{H}_2$ is naturally considered as the plane $\check{\mathbb{P}}^2$ dual to $\mathbb{P}^2$. Moreover, he showed, for one representation of F_4 as a power sum of linear forms $H_1, \dots, H_6$, the six points $[H_1], \dots, [H_6] \in \check{\mathbb{P}}^2$ correspond to six conics through one point of V_{22}.

Even if F_4 is taken as a special ternary quartic, $\mathrm{VSP}\,(F_4, 6)$ may be still a smooth prime Fano threefold of genus twelve. Mukai [Muk92, §7] shows that, if F_4 is the square of a non-degenerate quadratic form, then $\mathrm{VSP}\,(F_4, 6)$ is so called the Mukai-Umemura threefold discovered in [MU83] as a smooth $\mathrm{SO}(3, \mathbb{C})$-equivariant compactification of $\mathrm{SO}(3, \mathbb{C})/\mathrm{Icosa}$. N. Manolache and F.-O. Schreyer [MS01] and F. Melliez and K. Ranestad [MR05] show that, if F_4 is the Klein quartic, then $\mathrm{VSP}\,(F_4, 6)$ is a smooth compactification of the moduli space of $(1, 7)$-polarized abelian surfaces.

1.3. **Geometry of conics and lines and the main result.**
Our main result, given in the end of the section 2, is a generalization of Mukai's result Theorem 1.2.1; we describe certain subvarieties of the varieties of power sum of special quartic forms in any number $v + 1$ of variables. The quartics correspond to the ones of Theorem 1.2.1 if $v = 2$.

For this we generalize Mukai's study of the geometries of lines and conics on V_{22}. We recall Iskovskih's description of the so-called double projection of a V_{22} from a general line as follows:

$$\begin{tikzcd} & A' \arrow[dl, "f'"'] \arrow[rr, dashed] & & A \arrow[dr, "f"] & \\ V_{22} & & & & B, \end{tikzcd}$$

where

- f' is the blow-up along a general line,
- B is the smooth quintic del Pezzo threefold, namely, a smooth projective threefold such that $-K_B = 2H$, where H is the ample generator of $\mathrm{Pic}\, B$ and $H^3 = 5$, and
- f is the blow-up along a smooth rational curve of degree five (with respect to H).

Generalizing this situation we consider a general smooth rational curve of degree d on B, where d is an arbitrary integer greater than or equal to 5. In 2.2, we establish the existence of such a C and we study some of its properties, especially, the relations to lines and conics on B intersecting it. Let $f\colon A \to B$ be the blow up of B along C. In 2.3.2 and 2.4.2, we define lines and conics on A, which are appropriate generalizations of lines and conics on V_{22}. We say l is a *line* on A if l is a reduced connected curve with $-K_A \cdot l = 1$, $E_C \cdot l = 1$ and $p_a(l) = 0$, where $E_C := f^{-1}(C)$ is the exceptional divisor of $f\colon A \to B$. We say q is a *conic* on A if q is a reduced connected curve with $-K_A \cdot q = 2$, $E_C \cdot q = 2$ and $p_a(q) = 0$.

We see that lines on A are parameterized by a smooth trigonal canonical curve $\mathcal{H}_1$ of genus $d - 2$ (Corollary 2.3.1). Conics on A turn out to be parameterized by a smooth surface $\mathcal{H}_2$. The study of $\mathcal{H}_2$ is quite delicate. For this purpose, we

consider the intersection of lines and conics and introduce the divisor $D_l \subset \mathcal{H}_2$ parameterizing conics which intersect a fixed line l. We show that C has $\frac{(d-2)(d-3)}{2}$ bisecant lines and using this we can state the apparently simple result:

Theorem 1.3.1 (see Theorem 2.4.18). *The surface $\mathcal{H}_2$ which parameterizes conics on A is smooth and it is obtained by the blow-up $\eta\colon \mathcal{H}_2 \to S^2C \simeq \mathbb{P}^2$ at the points c_i where c_i is the point of S^2C corresponding to the intersection of the bisecants β_i and C, $i = 1, \dots, \frac{(d-2)(d-3)}{2}$.*

Moreover, we show that if $d \geq 6$, then $|D_l|$ is very ample and embeds $\mathcal{H}_2$ in $\check{\mathbb{P}}^{d-3}$, and if $d = 5$, $|D_l|$ defines a birational morphism $\mathcal{H}_2 \to \check{\mathbb{P}}^2$. Here we use the dual notation for later convenience. If $d \geq 6$, then $\mathcal{H}_2$ is so called the *White surface* (see [Whi24] and [Gim89]). It is interesting for us that the classical White surface naturally appears in this set up.

A deeper understanding of the geometry of conics requires the notion of intersection of two conics and, more precisely, the divisor $D_q \subset \mathcal{H}_2$ parameterizing conics which intersect a fixed conic q. It is easy to see that $D_q \sim 2D_l$.

Now assuming $d \geq 6$ we consider $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3} = \mathbb{P}_*\check{V}$. By the double projection of B from a general point b, we see that there are $n := \frac{(d-1)(d-2)}{2}$ conics (counted with multiplicities) through b. It is crucial that the number n is equal to the dimension of the quadratic forms on $\check{\mathbb{P}}^{d-3}$. Nevertheless infinitely many conics on A pass through a point on the strict transform of a bi-secant line of C. Hence to have a finiteness result we have to consider the blow-up $\rho\colon \widetilde{A} \to A$ along the strict transforms of bi-secant lines of C on B. Then by a careful analysis on mutually intersecting conics on A we construct a morphism $\Phi\colon \widetilde{A} \to \mathrm{Hilb}^n\check{\mathbb{P}}^{d-3}$ obtained by an attaching process which associates n conics on A to each point $\widetilde{a}$ of $\widetilde{A}$; see Definition 2.5.8 for the precise definition of attached conics. To produce the quartic we are looking for, we show that the proper locus $\{[q] \in \mathcal{H}_2 \mid [q] \in D_q\}$ on $\mathcal{H}_2$ is cut out by a quartic, whose equation is denoted by $\check{F}_4$. Moreover we show that $\check{F}_4$ is non-degenerate, this means that the polar map induced by $\check{F}_4$ from $S^2\check{V}$ to S^2V is an isomorphism. Then the required quartic F_4 is the dual quartic to $\check{F}_4$, namely, the quartic form in $S^4\check{V}$ such that its induced polar map from S^2V to $S^2\check{V}$ is the inverse of that of $\check{F}_4$.

For the precise statement of our main result, we need the following definition:

Definition 1.3.2. For a subvariety S of $\mathbb{P}_*\check{V}$, we set

$$\mathrm{VSP}\,(F, n; S) := \overline{\{([H_1], \dots, [H_n]) \mid [H_i] \in S, H_1^m + \cdots + H_n^m = F\}} \subset \mathrm{VSP}\,(F, n)$$

and we call it the *varieties of power sums of F confined in S*.

As far as we know, $\mathrm{VSP}\,(F, n; S)$ is essentially a new object to study.

Our main theoretical result is the following:

Theorem 1.3.3 (=Theorem 2.5.12). *There is an injection $\Phi\colon \widetilde{A} \to \mathrm{Hilb}^n\check{\mathbb{P}}^{d-3}$ mapping a point a of $\widetilde{A}$ to the point representing the n conics on A attached to a. Moreover the image is an irreducible component of* $\mathrm{VSP}\,(F_4, n; \mathcal{H}_2)$.

In the sequel 1.4, we explain a more significant geometrical meaning of the special quartic F_4.

Based on Mukai's result we can state the following conjecture: Φ is an embedding and $\mathrm{Im}\,\Phi = \mathrm{VSP}\,(F_4, n; \mathcal{H}_2)$.

We remark that, for $d \leq 8$, the number n is equal to the Waring rank of a general $(d-2)$-nary quartic, and especially, the cases where $d = 5, 6, 7$ cover exceptional cases of Alexander and Hirschowitz.

Even if $d = 5$, we have a similar result, which is an elaboration on Theorem 1.2.1. The explanation is technical: see 2.5.3.

1.4. **Applications.**

In the section 3, we give some applications of our study of A for a pair of a canonical curve of any genus and a non-effective theta characteristic, a *spin curve* for short.

Dolgachev and Kanev [DK93, §9] give a modern account of Scorza's beautiful construction of a certain quartic hypersurface, so called the *Scorza quartic*, associated to every spin curve. It is expected that the Scorza quartic is useful for the study of a spin curve but no deeper properties of the Scorza quartic were unknown. Firstly, its construction is not so explicit. Secondly, Scorza's construction itself depends on three assumptions on spin curves (see [DK93, (9.1) (A1)–(A3)]) and it were unknown whether these conditions are fulfilled for a general spin curve of genus > 3. Thus the existence of the Scorza quartic was conditional except for the genus 3 case, where Scorza himself solved the problem. We give contributions for these two subjects.

In 3.1, using the incidence correspondence on $\mathcal{H}_1 \times \mathcal{H}_1$ defined by intersections of lines on A, we define a non-effective theta characteristic θ on the trigonal curve $\mathcal{H}_1$. This is a generalization of Mukai's result explained as in 1.2.

In 3.2, we observe that there is a natural duality between $\mathcal{H}_1$ and $\mathcal{H}_2$, which induces the natural identification $\mathbb{P}^{d-3} = \mathbb{P}^* H^0(\mathcal{H}_1, K_{\mathcal{H}_1})$, where for clarity reasons we denote by $\mathbb{P}^{d-3}$ the projective space dual to the ambient projective space $\check{\mathbb{P}}^{d-3}$ of $\mathcal{H}_2$, and by $\mathbb{P}^* H^0(\mathcal{H}_1, K_{\mathcal{H}_1})$ the ambient projective space of the canonical embedding of $\mathcal{H}_1$.

In 3.3, we recall the definition of the discriminant loci and we compute it explicitly for $(\mathcal{H}_1, \theta)$. In 3.4, we recall the precise definition of the Scorza quartic for a spin curve.

By virtue of our explicit computation of the discriminant, we prove in 3.5 that the pair $(\mathcal{H}_1, \theta)$ satisfies the conditions [DK93, (9.1) (A1)–(A3)], which guarantee the existence of the Scorza quartic for the pair $(\mathcal{H}_1, \theta)$. Then, by a standard deformation theoretic argument, we can then verify that the conditions (A1)–(A3) hold also for a general spin curve, hence we answer affirmatively to the Dolgachev-Kanev Conjecture:

Theorem 1.4.1 (=Theorem 3.5.3)**.** *The Scorza quartic exists for a general spin curve.*

Moreover we can find explicitly the Scorza quartic for $(\mathcal{H}_1, \theta)$. In fact, by definition, the Scorza quartic $\{F'_4 = 0\}$ for $(\mathcal{H}_1, \theta)$ lives in $\mathbb{P}^* H^0(\mathcal{H}_1, K_{\mathcal{H}_1})$ but, as we remark above, we can consider $\{F'_4 = 0\} \subset \mathbb{P}^{d-3}$. In 3.6, we prove that the special quartic $\{F_4 = 0\} \subset \mathbb{P}^{d-3}$ in Theorem 1.3.3 coincides with the Scorza quartic $\{F'_4 = 0\}$.

We recommend the readers who are interested only in the subsections 3.1–3.5 to skip the subsection 2.5.

Finally, in 3.7, we show that A is recovered from the pair $(\mathcal{H}_1, \theta)$. This implies that $(\mathcal{H}_1, \theta)$'s fill up an open subset of the moduli of trigonal spin curves. In particular, $\mathcal{H}_1$ is a general trigonal curve for a general C.

1.5. **Final remarks.**

In this paper, we only consider a general rational curve on B but there are interesting special cases. In the forthcoming paper, applying the method of this paper, we will study the blow-ups of B along special smooth rational curves of degree six and pairs of canonical curves of genus four and even theta characteristics.

Acknowledgment. We are thankful to Professor S. Mukai for valuable discussions and constant interest on this paper. We received various useful comments from K. Takeuchi, A. Ohbuchi, S. Kondo, to whom we are grateful. The first author worked on this paper partially while he was working at the Research Institute for Mathematical Sciences, Kyoto University until March, 2004, and when he was staying at the Johns Hopkins University under the program of Japan-U.S. Mathematics Institute (JAMI) in November 2005 and at the Max-Planck-Institut für Mathematik from April, 2007 until March, 2008. Besides he gave talks on this paper in the seminars and the conferences at the Waseda University, the Tohoku University, the Johns Hopkins University, the Princeton University, and Mathematisches Forschungsinstitut Oberwolfach. The second author gave a talk on this paper in the seminar at the S.I.S.S.A. (Trieste) on May 2007 and at the Workshop on Hd. MMP. December 2007 (Warwick). The authors worked jointly during the first author's stay at the Università di Udine on August 2005, and the Levico Terme conference on Algebraic Geometry in Higher dimensions on June 2007. The authors are thankful to all institutes, and organizers of the seminars for the warm hospitality they received. Both authors would like to dedicate this paper to Miles Reid with full gratitude.

2. Blowing up the quintic del Pezzo threefold B along a $\mathbb{P}^1$ of degree d

In this section, we study the geometries of the blow-up A of the quintic del Pezzo threefold B along a smooth rational curve of degree d, which is nothing but the special threefold we mention in the abstract. In 2.1 we review the description of lines and conics on B, and 2-ray games originating from B. Based on this, we construct in 2.2 smooth rational curves of degree d, where d is an arbitrary positive integer, having nice intersection properties with respect to lines and conics. The results in 2.2 are delicate but their proof is more or less based on standard parameter count. In 2.3 and 2.4, we study the families of curves on A of degree one or two with respect to the anti-canonical sheaf of A (we call them *lines* and *conics* on A respectively). The curve $\mathcal{H}_1$ parameterizing lines on A and the surface $\mathcal{H}_2$ parameterizing conics on A are two of the main characters in this paper. See Corollary 2.3.1 and Theorem 2.4.18 for a quick view of their properties. Finally in 2.5, we prove the main theorem (Theorem 2.5.12). See 2.5.3 for the relationship of our result with Mukai's one we mentioned in the introduction.

2.1. **Review on geometries of B.**

Let V be a vector space with $\dim_{\mathbb{C}} V = 5$. The Grassmannian $G(2, V)$ embeds into $\mathbb{P}^9$ and we denote the image by $G \subset \mathbb{P}^9$. It is well-known that the quintic del

Pezzo 3-fold, i.e., the Fano 3-fold B of index 2 and of degree 5 can be realized as $B = G \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to G (see [Fuj81], [Isk77, Thm 4.2 (iii), the proof p.511-p.514]).

Let $\mathcal{H}_1^B$ and $\mathcal{H}_2^B$ be the Hilbert scheme, respectively, of lines and of conics on B. We collect basic known facts on lines and conics on B almost without proof.

2.1.1. *Lines on B.* Let $\pi\colon \mathbb{P} \to \mathcal{H}_1^B$ be the universal family of lines on B and $\varphi\colon \mathbb{P} \to B$ the natural projection. By [FN89a, Lemma 2.3 and Theorem I], $\mathcal{H}_1^B$ is isomorphic to $\mathbb{P}^2$ and φ is a finite morphism of degree three. In particular the number of lines passing through a point is three counted with multiplicities. We recall some basic facts about π and φ which we use in the sequel.

Before that, we fix some notation.

Notation 2.1.1. For an irreducible curve C on B, denote by $M(C)$ the locus $\subset \mathbb{P}^2$ of lines intersecting C, namely, $M(C) := \pi(\varphi^{-1}(C))$ with reduced structure. Since φ is flat, $\varphi^{-1}(C)$ is purely one-dimensional. If $\deg C \geq 2$, then $\varphi^{-1}(C)$ does not contain a fiber of π, thus $M(C)$ is a curve. See Proposition 2.1.3 for the description of $M(C)$ in case C is a line.

Definition 2.1.2. A line l on B is called a *special line* if $\mathcal{N}_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$.

Proposition 2.1.3. *It holds*:

(1) *for the branched locus B_φ of $\varphi\colon \mathbb{P} \to B$ we have:*
  (1-1) *$B_\varphi \in |-K_B|$,*
  (1-2) *$\varphi^* B_\varphi = R_1 + 2R_2$,*
  (1-3) *$R_1 \simeq R_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and*
  (1-4) *$\varphi\colon R_1 \to B_\varphi$ and $\varphi\colon R_2 \to B_\varphi$ are injective,*

(2) *R_2 is contracted to a conic Q_2 by $\pi\colon \mathbb{P} \to \mathcal{H}_1^B$. Moreover Q_2 is the branched locus of $\pi_{|R_1}\colon R_1 \to \mathcal{H}_1^B$,*

(3) *Q_2 parameterizes special lines. If l is not a special line on B, then $\mathcal{N}_{l/B} = \mathcal{O}_l \oplus \mathcal{O}_l$,*

(4) *if l is a special line, then $M(l)$ is a line, and $M(l)$ is tangent to Q_2 at $[l]$. If l is not a special line, then $M(l)$ is the disjoint union of a line and the point $[l]$.* By abuse of notation, *we denote by $M(l)$ the one-dimensional part of $M(l)$ for any line l. Vice-versa, any line in $\mathcal{H}_1^B$ is of the form $M(l)$ for some line l,*

(5) *the locus swept by lines intersecting l is a hyperplane section T_l of B whose singular locus is l. For every point b of $T_l \setminus l$, there exists exactly one line which belongs to $M(l)$ and passes through b. Moreover, if l is not special, then the normalization of T_l is $\mathbb{F}_1$ and the inverse image of the singular locus is the negative section of $\mathbb{F}_1$, or, if l is special, then the normalization of T_l is $\mathbb{F}_3$ and the inverse image of the singular locus is the union of the negative section and a fiber, and*

(6) *if l is not a special line, then $\varphi^{-1}(l)$ is the disjoint union of the fiber of π corresponding to l, and the smooth rational curve dominating $M(l)$.*

Proof. See [FN89a, §2] and [Ili94, §1]. □

By the proof of [FN89a] we see that B is stratified according to the ramification of $\varphi\colon \mathbb{P} \to B$ as follows:

$$B = (B \setminus B_\varphi) \cup (B_\varphi \setminus C_\varphi) \cup C_\varphi,$$

where C_φ is a smooth rational normal sextic and if $b \in B \setminus B_\varphi$ exactly three distinct lines pass through it, if $b \in (B_\varphi \setminus C_\varphi)$ exactly two distinct lines pass through it, one of them is special, and finally C_φ is the loci of $b \in B$ through which it passes only one line.

2.1.2. *Conics on B.*

Proposition 2.1.4. *The Hilbert scheme of conics on B is isomorphic to $\mathbb{P}^4 = \mathbb{P}_*\check{V}$. The support of a double line is a special line and the double lines are parameterized by a rational normal quartic curve $\Gamma \subset \mathbb{P}_*\check{V}$ and the secant variety of Γ is a singular cubic hypersurface which is the closure of the loci parameterizing reducible conics.*

Proof. See [Ili94, Proposition 1.2.2]. □

The identification is given by the map $sp\colon \mathcal{H}_2^B \to \mathbb{P}_*\check{V}$ with $[c] \mapsto \langle Gr(c)\rangle = \mathbb{P}^3_c \subset \mathbb{P}_*V$, where for a general conic $c \subset B$ we set

$$Gr(c) := \cup\{r \in \mathbb{P}_*V \mid [r] \in c\} \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

2.1.3. *Two-ray games based on B.* We are interested in the geometry of the 3-fold A obtained by blowing-up B along a curve $C := C_d$ as constructed in Proposition 2.2.4. To understand this geometry we need to describe some two-ray games originating from B.

Definition 2.1.5. Let b be a point of B. We call the rational map from B defined by the linear system of hyperplane sections singular at b *the double projection from b*.

Proposition 2.1.6. (1) *Let b be a point of B. Then the target of the double projection from b is $\mathbb{P}^2$, and the double projection from b and the projection $B \dashrightarrow \overline{B}_b$ from b fit into the following diagram:*

(2.1)
$$\begin{tikzcd} & B_b \arrow[dl, "\pi_{1b}"'] \arrow[dr] & & B'_b \arrow[dl] \arrow[dr, "\pi_{2b}"] & \\ B & & \overline{B}_b & & \mathbb{P}^2, \end{tikzcd}$$

where π_{1b} is the blow-up of B at b, $B_b \dashrightarrow B'_b$ is the flop of the strict transforms of lines through b, and $\pi_{2b}\colon B'_b \to \mathbb{P}^2$ is a (unique) $\mathbb{P}^1$-bundle structure. We denote by E_b the π_{1b}-exceptional divisor and by E'_b the strict transform of E_b on B'_b. Moreover we have the following descriptions:

(1-1)
$$-K_{B'_b} = H + L,$$
where H is the strict transform of a general hyperplane section of B, and L is the pull back of a line on $\mathbb{P}^2$,

(1-2) *if $b \notin B_\varphi$, then the strict transforms l'_i of three lines l_i through b on B_b have the normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The flop $B_b \dashrightarrow B'_b$ is the Atiyah flop. In particular, $E'_b \to E_b$ is the blow-up at the three points $E_b \cap l'_i$.*

If $b \in B_\varphi \setminus C_\varphi$, then $E_b \dashrightarrow E'_b$ can be described as follows: *let l and m be two lines through b, where l is special, and m is not special. Let l' and m' be the strict transforms of l and m on B_b. First blow up E_b at two*

points $t_1 := E_b \cap l'$ *and* $t_2 := E_b \cap m'$ *and then blow up at a point* t_3 *on the exceptional curve* e *over* t_1. *Finally, contract the strict transform of* e *to a point. Then we obtain* E'_b (*this is a degeneration of the case* (a)). *See* [FN89b] *in case of* $b \in C_\varphi$, *and*

(1-3) *a fiber of* π_{2b} *not contained in* E'_b *is the strict transform of a conic through* b, *or the strict transform of a line* $\not\ni b$ *intersecting a line through* b. *The description of the fibers of* π_{2b} *contained in* E'_b *is as follows*: *if* $b \notin B_\varphi$, *then* $\pi_{2b|E'_b} \colon E'_b \to \mathbb{P}^2$ *is the blow-down of the strict transforms of three lines connecting two of* $E_b \cap l'_i$, *namely,* $E_b \dashrightarrow \mathbb{P}^2$ *is the Cremona transformation.*
Assume that $b \in B_\varphi \setminus C_\varphi$. *Then* $\pi_{2b|E'_b} \colon E'_b \to \mathbb{P}^2$ *is the blow-down of the strict transforms of two lines, one is the line connecting* t_1 *and* t_2, *the other is the line whose strict transform passes through* t_3. $E_b \dashrightarrow \mathbb{P}^2$ *is a degenerate Cremona transformation. See* [FN89b] *in case of* $b \in C_\varphi$.

(2) *Let* l *be a line on* B. *Then the projection of* B *from* l *is decomposed as follow*:

$$\begin{tikzcd} & B_l \arrow[dl, "\pi_{1l}"'] \arrow[dr, "\pi_{2l}"] & \\ B & & Q, \end{tikzcd} \tag{2.2}$$

where π_{1l} *is the blow-up along* l *and* $B \dashrightarrow Q$ *is the projection from* l *and* π_{2l} *contracts onto a rational normal curve of degree* 3 *the strict transform of the loci swept by the lines of* B *touching* l. *Moreover*

$$-K_{B_l} = H + H_Q, \tag{2.3}$$

where H *and* H_Q *are the pull backs of general hyperplane sections of* B *and* Q *respectively. We denote by* E_l *the* π_{1l}*-exceptional divisor.*

(3) *Let* q *be a smooth conic on* B. *Then the projection of* B *from* q *behaves as follow*:

$$\begin{tikzcd} & B_q \arrow[dl, "\pi_{1q}"'] \arrow[dr, "\pi_{2q}"] & \\ B & & \mathbb{P}^3, \end{tikzcd} \tag{2.4}$$

where π_{1q} *is the blow-up of* B *along* q *and* $\pi_{2q} \colon B_q \to \mathbb{P}^3$ *is the divisorial contraction of the strict transform* T_q *of the loci swept by the lines touching* q. *Moreover*

$$-K_{B_q} = H + H_{\mathbb{P}}, \tag{2.5}$$

where H *and* $H_{\mathbb{P}}$ *are the pull backs of general hyperplane sections of* B *and* $\mathbb{P}^3$ *respectively.*

Proof. These results come from explicit computations and are more or less known. Especially, for (2), refer [Fuj81], and for (3) (and (2)), refer [MM81], No. 22 for (3) (No. 26 for (2)). See also [MM85], p.533 (7.7) for a discussion.

(1) is less known. We have only found the paper [FN89b], in which they deal with the most difficult case (c). Here we only sketch the construction of the flop in the middle case (b) to intend the reader to get a feeling of birational maps from B.

Let b be a point of $B_\varphi \setminus C_\varphi$. We use the notation of the statement of (1-2). The flop of m' is the Atiyah flop. We describe the flop of l'. By $\mathcal{N}_{l/B} \simeq \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, it holds that $\mathcal{N}_{l'/B_b} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. Hence the flop of l' is a special case of Reid's one [Rei83, Part II]. We show that the width is two in Reid's sense. Let T_1 be the normalization of T_l. By Proposition 2.1.3 (5), $T_1 \simeq \mathbb{F}_3$ and the inverse image of the singular locus of T_l is the union of the negative section C_0 and a fiber r. Let $\mu\colon \widetilde{B}_b \to B_b$ be the blow-up along l' and F the exceptional divisor. Let T_2 be the strict transform of T_l on $\widetilde{B}_b$. Then T_2 is the blow-up of T_1 at two points $s_1 \in C_0$ and $s_2 \in r$. Denote by C_0' and r' the strict transforms of C_0 and r. We prove that $\mathcal{N}_{r'/\widetilde{B}_b} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$. Note that $F \cap T_2 = C_0' \cup r'$. The curves C_0' and r' are two sections on F. Let T_1' be the image of T_2 on B_b. By $\mathcal{N}_{l'/B_b} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and $T_2 = \mu^* T_1' - 2F$, it holds $F \simeq \mathbb{F}_2$, and $T_{2|F} \sim 2G_0 + 3\gamma$, where G_0 is the negative section of F and γ is a fiber of $F \to l'$. Note that $F \cdot C_0' = (F_{|T_2} \cdot C_0')_{T_2} = -3$ and $F \cdot r' = (F_{|T_2} \cdot r')_{T_2} = 0$, and $F \cdot G_0 = 0$ and $F \cdot (G_0 + 3\gamma) = -3$. Thus we have $C_0' \sim G_0 + 3\gamma$ and $r' = G_0$ on F. Now we see that $-K_{\widetilde{B}_b} \cdot r' = (\mu^*(-K_{B_b}) - F) \cdot r' = 0$. Therefore, by $(r')^2 = -1$ on T_2, it holds that $\mathcal{N}_{r'/\widetilde{B}_b} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$.

It is easy to see that we can flop r'. Let $\widetilde{B}_b \dashrightarrow \widetilde{B}_b'$ be the flop of r' (now we consider locally around r'). Let F' be the strict transform of F on $\widetilde{B}_b'$. By [Rei83], $F' \simeq F$ and there is a blow-down $\widetilde{B}_b' \to \widetilde{B}_b''$ of F' such that $\widetilde{B}_b''$ is smooth. $\widetilde{B}_b \dashrightarrow \widetilde{B}_b''$ is the flop of l'.

By this description of the flop, we can easily obtain (1-2). □

As a first application of the above operations, we have the following result, which we often use:

Corollary 2.1.7. *Let b_1 and b_2 be two (possibly infinitely near) points on B such that there exists no line on B through them. Then there exists a unique conic on B through b_1 and b_2.*

Proof. We project B from b_1 as in (2.1). Then the assertion follows by the description of fibers of π_{2b_1} as in Proposition 2.1.6 (1-3). □

2.2. Construction of smooth rational curves C_d of degree d on B.

We construct smooth rational curves of degree d on B with certain properties.

Proposition 2.2.1. *There exists a smooth rational curve C_d of degree d on B such that*

(a) *a general line on B intersecting C_d is uni-secant,*
(b) *C_d is obtained as a smoothing of the union of a smooth rational curve C_{d-1} of degree $d-1$ on B and a general uni-secant line of it on B,*
(c) *$\mathcal{N}_{C_d/B} \simeq \mathcal{O}_{\mathbb{P}^1}(d-1) \oplus \mathcal{O}_{\mathbb{P}^1}(d-1)$. In particular $h^1(\mathcal{N}_{C_d/B}) = 0$ and $h^0(\mathcal{N}_{C_d/B}) = 2d$, and*
(d) *if $d = 5$, then C_5 is a normal rational curve and is contained in a unique hyperplane section S, which is smooth. If $d \geq 6$, then C_d is not contained in a hyperplane section.*

Proof. We argue by induction on d.

If $d = 1$, we have the assertion since $\mathcal{N}_{C_1/Q} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ for a general line C_1.

Now assume that C_{d-1} is a general smooth rational curve of degree $d-1$ on B. By induction, a general secant line l of C on Q is uni-secant. Set $Z := C_{d-1} \cup l$ and $\mathcal{N}_{Z/Q} := \mathcal{H}om_{\mathcal{O}_Q}(\mathcal{I}_Z, \mathcal{O}_Q)$. By induction, the normal bundle of C_{d-1} satisfies (c).

Thus, by $\mathcal{N}_{l/Q} \simeq \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ and [HH85, Theorem 4.1 and its proof], it holds $h^1(\mathcal{N}_{Z/Q}) = 0$, and moreover $Z := C_{d-1} \cup l$ is strongly smoothable, namely, we can find a smoothing C_d of Z with the smooth total space. By the upper semi-continuity theorem, we have $h^1(\mathcal{N}_{C_d/Q}) = 0$.

Thus the Hilbert scheme of Q is smooth at $[C_d]$ and is of dimension $2d$, which is also the dimension of the component of the Hilbert scheme containing $[C_d]$. On the other hand, varying C_{d-1} and l, we have a family of reducible curves of dimension $2(d-1)+1 = 2d-1$. Thus the smoothing constructed as above is general in the component of the Hilbert scheme whose generic point parameterizes smooth rational curves of degree d.

It is easy to see that a general line m intersecting C_{d-1} does not intersect l, thus m is a uni-secant line of $C_{d-1} \cup l$. This implies (a) for C_d by a deformation theoretic argument.

To check the form of the normal bundle, simply assume by induction that $\mathcal{N}_{C_{d-1}/Q} = \mathcal{O}_{\mathbb{P}^1}(d-2) \oplus \mathcal{O}_{\mathbb{P}^1}(d-2)$. Set $\mathcal{N}_{C_d/Q} := \mathcal{O}_{\mathbb{P}^1}(a_d) \oplus \mathcal{O}_{\mathbb{P}^1}(b_d)$ $(a_d \geq b_d)$ for the smoothing C_d of Z. We show that $a_d = b_d = d-1$.

It suffices to prove $h^0(\mathcal{N}_{Z/Q}(-d)) = 0$. In fact, then, by the upper semi-continuous theorem, we have $h^0(\mathcal{N}_{C_d/Q}(-d)) = 0$ and $a_d, b_d \leq d-1$. Thus, by $a_d + b_d = 2d-2$, it holds $a_d = b_d = d-1$.

The equality $h^0(\mathcal{N}_{Z/Q}(-d)) = 0$ easily follows from the following three exact sequences, where $t := C_{d-1} \cap l$:

$$0 \to \mathcal{N}_{Z/Q} \to \mathcal{N}_{Z/Q|C_{d-1}} \oplus \mathcal{N}_{Z/Q|l} \to \mathcal{N}_{Z/Q} \otimes_{\mathcal{O}_Q} \mathcal{O}_t \to 0.$$

$$0 \to \mathcal{N}_{C_{d-1}/Q} \to \mathcal{N}_{Z/Q|C_{d-1}} \to T^1(t) \to 0.$$

$$0 \to \mathcal{N}_{l/Q} \to \mathcal{N}_{Z/Q|l} \to T^1(t) \to 0.$$

Finally we prove (d). In case $d = 5$, we have only to notice that a general hyperplane section of B_5 is a del Pezzo surface of degree 5 which contains a smooth C_5. For $d \geq 6$, the assertion follows by induction. □

We denote by $\mathcal{H}_d^B$ the union of components of the Hilbert scheme of B whose general points parameterize smooth rational curves of degree d obtained inductively as in Proposition 2.2.1.

The following proposition describe relations between C_d and lines and conics on B.

Proposition 2.2.2. *A general C_d as in Proposition 2.2.1 satisfies the following conditions*:

(1) *there exist no k-secant lines of C_d on B with $k \geq 3$,*
(2) *there exist at most finitely many bi-secant lines of C_d on B, and any of them intersects C_d simply,*
(3) *bi-secant lines of C_d on B are mutually disjoint,*
(4) *neither a bi-secant line nor a line through the intersection point between a bi-secant line and C_d is a special line,*

(5) *there exist at most finitely many points b outside C_d such that all the lines through b intersect C_d, and such points exist outside bi-secant lines of C_d,*
(6) *there exist no k-secant conics of C_d with $k \geq 5$,*
(7) *there exist at most finitely many quadri-secant conics of C_d on B, and no quadri-secant conic is tangent to C_d, and*
(8) *$q_{|C_d}$ has no point of multiplicity greater than two for any multi-secant conic q.*

Proof. We can prove the assertions by simple dimension count based upon Proposition 2.2.1. We assume that $d \geq 4$ since otherwise we can verify the assertion easily.

(1). Let $\mathcal{D}$ be the closure of the set

$$\{([C_d],[l]) \mid C_d \cap l \text{ consists of 3 points}\} \subset \mathcal{H}_d^B \times \mathcal{H}_1^B.$$

Let $\pi_d \colon \mathcal{D} \to \mathcal{H}_d^B$ and $\pi_1 \colon \mathcal{D} \to \mathcal{H}_1^B$ be the natural morphisms induced by the projections. The claim follows if we show that $\dim_{\mathbb{C}} \mathcal{D} \leq 2d-1$ since $\dim \mathcal{H}_d^B = 2d$.

Thus we estimate $\dim_{\mathbb{C}} \operatorname{Hom}^{2d}(\mathbb{P}^1, B; p_i \mapsto s_i, i=1,2,3)$ at $[\pi]$, where p_i, $i = 1,2,3$ are fixed points of $\mathbb{P}^1$, $[\pi]$ is a general point and the degree is measured by $-K_B$. By $d \geq 4$ and Proposition 2.2.1 (c), it holds that $h^0(\mathbb{P}^1, \pi^*T_B(-p_1-p_2-p_3)) = 2d-6$ and $h^1(\mathbb{P}^1, \pi^*T_B(-p_1-p_2-p_3)) = 0$. Then

$$\dim_{\mathbb{C}} \operatorname{Hom}^{2d}(\mathbb{P}^1, B, p_i \mapsto s_i, i=1,2,3)_{[\pi]} = h^0(\pi^*T_B(-p_1-p_2-p_3)) = 2d-6.$$

This implies that $\dim_{\mathbb{C}} \pi_1^{-1}([l]) \leq 2d-6+3 = 2d-3$ since the three points can be chosen arbitrarily. Then $\dim_{\mathbb{C}} \mathcal{D} \leq 2d-1$ since $\dim_{\mathbb{C}} \mathcal{H}_1^B = 2$.

(2). Now let $\mathcal{D}$ be the closure of the set

$$\{([C_d],[l]) \mid C_d \cap l \text{ consists of 2 points}\} \subset \mathcal{H}_d^B \times \mathcal{H}_1^B.$$

As before, let $\pi_d \colon \mathcal{D} \to \mathcal{H}_d^B$ and $\pi_1 \colon \mathcal{D} \to \mathcal{H}_1^B$ be the natural morphisms induced by the projections. By $d \geq 4$ and Proposition 2.2.1 (c), it holds that $h^0(\mathbb{P}^1, \pi^*T_B(-p_1-p_2)) = 2d-3$ and $h^1(\mathbb{P}^1, \pi^*T_B(-p_1-p_2)) = 0$. Then

$$\dim_{\mathbb{C}} \operatorname{Hom}^{2d}(\mathbb{P}^1, B, p_i \mapsto s_i, i=1,2)_{[\pi]} = h^0(\pi^*T_B(-p_1-p_2)) = 2d-3.$$

Since $\dim_{\mathbb{C}} \operatorname{Aut}(\mathbb{P}^1, p_1, p_2) = 1$ it holds that $\dim_{\mathbb{C}} \pi_1^{-1}([l]) \leq 2d-3+2-1 = 2d-2$. Hence $\dim_{\mathbb{C}} \mathcal{D} = 2d$. Hence C_d has only a finite number of bisecant lines.

We now show that the loci where C_d has a tangent bisecant is a codimension one loci inside $\mathcal{H}_d^B$. Let B_t be the blow-up of B in a point $t \in C_d$ and let l be a bi-secant which is tangent to C_d at t (if it exists). Let E be the exceptional divisor, and C' and l' the strict transforms of C and l respectively. By hypothesis there exists a unique point $s \in E \cap C' \cap l'$. We estimate $\dim_{\mathbb{C}} \operatorname{Hom}^{d-2}(\mathbb{P}^1, B_t, p \mapsto s)_{[\pi]}$, where p is a fixed point of $\mathbb{P}^1$, $[\pi]$ is a general point, and the degree is measured by $-K_{B_t}$. In this case $h^0(\pi^*T_{B_t}(-p)) = 2d-2$ hence $\dim_{\mathbb{C}} \pi_1^{-1}([l]) \leq 2d-2+1-2 = 2d-3$. This implies the claim.

The cases (3), (4) and (5) are similar. Thus we only give few comments for (5). Set $\mathcal{D}$ be the closure of the set

$$\begin{aligned}\{([C_d],[l_1],[l_2],[l_3]) \mid\ & C_d \cap l_i \neq \emptyset\, (i=1,2,3),\\ & l_1 \cap l_2 \cap l_3 \neq \emptyset, l_1 \cap l_2 \cap l_3 \not\in C_d, l_i \text{ are distinct}\}\\ & \subset \mathcal{H}_d^B \times \mathcal{H}_1^B \times \mathcal{H}_1^B \times \mathcal{H}_1^B.\end{aligned}$$

For the former half of (5), we have only to prove that $\dim \mathcal{D} \leq 2d$. This can be carried out by a similar dimension count as above. For the latter half of (5), we use

the inductive construction of C_d besides dimension count. We can omit the proof of (6)–(8) since are definitely similar to those of (1)–(3).

$\square$

Notation 2.2.3. Denote by β_i $(i = 1, \cdots, s)$ bi-secant lines of C_d.

In the following proposition, we describe some more relations between C_d and lines on B by using $M(C_d) \subset \mathcal{H}_1^B$.

Proposition 2.2.4. *A general C_d as in Proposition 2.2.1 satisfies the following conditions*:

(1) *C_d intersects B_φ simply,*
(2) *$M_d := M(C_d)$ intersects Q_2 simply,*
(3) *M_d is an irreducible curve of degree d with only simple nodes (recall that in Proposition 2.1.3, we abuse the notation by denoting the one-dimensional part of $\pi(\varphi^{-1}(C_1))$ by $M(C_1)$),*
(4) *for a general line l intersecting C_d, $M_d \cup M(l)$ has only simple nodes as its singularities, and*
(5) *$M_d \cup M(\beta_i)$ has only simple nodes as its singularities.*

Proof. We show the assertion inductively using the smoothing construction of C_d from the union of C_{d-1} and a general uni-secant line l of C_{d-1}.

In case of $d = 1$, by letting C_1 be a general line, the assertion follows from Proposition 2.1.3. By induction on d assume that we have a smooth C_{d-1} $(d \geq 2)$ satisfying (1)–(5). We verify $C_{d-1} \cup l$ satisfies the following (1)'–(5)', which are suitable modifications of (1)–(5):
(1)' $C_{d-1} \cup l$ intersects B_φ simply by (1) for C_{d-1} and generality of l.
(2)' $M_{d-1} \cup M(l)$ intersects Q_2 simply by (2) for C_{d-1} and generality of l.
(3)' $M_{d-1} \cup M(l)$ is not irreducible but is of degree d and has only simple nodes by (4) for C_{d-1}.
(4)' $M_{d-1} \cup M(l) \cup M(m)$ has only simple nodes as its singularities for a general line m intersecting C_{d-1}.

Indeed, since m is also general, $M_{d-1} \cup M(m)$ has only simple nodes by (4) for C_{d-1}. Thus we have only to prove that $M_{d-1} \cap M(l) \cap M(m) = \emptyset$, namely, there is no secant line of C_{d-1} intersecting both l and m. Fix a general l and move m. If there are secant lines r_m of C_{d-1} intersecting both l and m for general m's, then r_m moves whence we have $M(l) \subset M_{d-1}$, a contradiction.
(5)' For a bi-secant line β of $C_{d-1} \cup l$ except the lines through $C_{d-1} \cap l$, the curve $M_{d-1} \cup M(l) \cup M(\beta)$ has only simple nodes as its singularities.

Indeed, if β is a bi-secant line of C_{d-1}, then the assertion follows from (5) for C_{d-1} by a similar way to the proof of (4)'. Suppose that β is a uni-secant line of C_{d-1} intersecting l. We have only to prove that there is no secant line of C intersecting both l and β. If there is such a line r, then l, β and r pass through one point. This does not occur for general l and β by Proposition 2.2.2 (5).

Thus, by a deformation theoretic argument, we see that C_d satisfies (1)–(5). $\square$

Notation 2.2.5. Consider the double projection from b, see proposition 2.1.6 [(1)]. Throughout the paper, we denote by C'_b, C''_b and C_b the strict transforms of $C := C_d$ on B_b, B'_b and $\mathbb{P}^2$ respectively.

The following result is one of the key properties of the component $\mathcal{H}_d^B$. Its importance and difficulty lies in the actual fact that it holds for every $b \in B$.

Proposition 2.2.6. *Let C_d be a general smooth rational curve of degree d on B constructed as in Proposition 2.2.1. Then, for any point $b \in B$, the restriction of π_b to C_d is birational if $d \geq 5$.*

Proof. We prove the assertion by induction based on the construction of C_d from $C_{d-1} \cup l$, where l is a general uni-secant line of C_{d-1} on B.

Assume that $d = 5$ and $\pi_{b|C_5}$ is not birational for a b. Then C_b is a line or conic in $\mathbb{P}^2$. Let S be the pull-back of C_b by π_{2b}. If C_b is a line, then C_5 is contained in a singular hyperplane section, which is the strict transform of S on B (recall that $B \dashrightarrow \mathbb{P}^2$ is the double projection from b). This contradicts Proposition 2.2.1 (d). Assume that C_b is a conic.

The only possibility is that $L \cdot C''_b = 4$ and $C''_b \to C_b$ is a double cover since $\deg C_b \cdot \deg(C''_b \to C_b) \leq 5$. By Proposition 2.1.6 (1-1), it holds $H \cdot C''_b = 6$. Then by $L = H - 2E'_b$, we have $E'_b \cdot C''_b = 1$. Note that ${E'_b}^2 S = 2$. On S, we can write $E'_{b|S} \sim C_0 + pl$ and $C''_b \sim 2C_0 + ql$ $(p, q \geq 0)$, where C_0 is the negative section of S and l is a fiber of $S \to C''_b$. Set $e := -C_0^2$. By $E'_b \cdot C''_b = 1$ and ${E'_b}^2 S = 2$, we have $q + 2p - 2e = 1$ and $2p - e = 2$. Thus $e = 2p - 2$ and $q = 2p - 3$. Since C''_b is irreducible, $q \geq 2e$, whence $2p - 3 \geq 2(2p - 2)$, i.e., $p = 0$ and $q = -3$, a contradiction.

Assume that $d \geq 6$. Let $\mathcal{C} \to \Delta$ be the one-parameter smoothing of $C_{d-1} \cup l$ such that $\mathcal{C}$ is smooth. We consider the trivial family of the double projections $B \times \Delta \dashrightarrow \mathbb{P}^2 \times \Delta$ from $b \times \Delta$. Denote by $\mathcal{C}'_b, \mathcal{C}''_b$ and $\mathcal{C}_b$ the strict transforms of $\mathcal{C}$ on $B'_b \times \Delta$, $B''_b \times \Delta$ and $\mathbb{P}^2 \times \Delta$ respectively. We also denote by $C'_{d-1,b}$, $C''_{d-1,b}$, and $C_{d-1,b}$ the strict transforms of C_{d-1} on B'_b, B''_b and $\mathbb{P}^2$ respectively. It suffices to prove that there exists at least one point on C_{d-1} where $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational.

Indeed, set

$$\mathcal{N} := \{(b, t) \in B \times \Delta \mid \mathcal{C} \dashrightarrow \mathcal{C}_b \text{ is not birational at any point of } \mathcal{C}_t\}$$

and let $\Delta' \subset \Delta$ be the image of $\mathcal{N}$ by the projection to Δ. $\mathcal{N}$ is a closed subset, and so is Δ' since $B \times \Delta \to \Delta$ is proper. Thus Δ' consists of finitely many points since the origin is not contained in Δ'. For a point $t \in \Delta$ sufficiently near the origin, $\mathcal{C}_t \dashrightarrow \mathcal{C}_{t,b}$ is birational for any b.

By induction, we may assume that $C_{d-1} \dashrightarrow C_{d-1,b}$ is birational. Note that $C_{d-1,b}$ is not a line since otherwise C_{d-1} is contained in a singular hyperplane section as we see above in the case of C_5, a contradiction. As for l, if $b \notin l$, then the image of l is a line or a point on $\mathbb{P}^2$. If $b \in l$, then the strict transform of l on B_b is a flopping curve. Thus $\mathcal{C}_b$ contains the line corresponding to l. We investigate the other possible irreducible components of the central fiber $\mathcal{C}_{b,0}$ of $\mathcal{C}_b \to \Delta$. If $b \notin C_{d-1} \cup l$, then the only possibility is that $\mathcal{C}_{b,0}$ contains the image of a flopped curve, which is a line on $\mathbb{P}^2$. Thus $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of C_{d-1}. If $b \in C_{d-1} \cup l$, then $\mathcal{C}_{b,0}$ contains the image m_b of the strict transform m''_b of a line m'_b in E_b through $E_b \cap (C'_{d-1,b} \cup l'_b)$, where l'_b is the strict transform of l on B_b. The line m'_b is nothing but the exceptional curve for $\mathcal{C}'_b \to \mathcal{C}$ (recall that $\mathcal{C}$ is a smooth surface). Moreover, if $b \in l$, then by the description of $E_b \dashrightarrow \mathbb{P}^2$, m_b is a line since l_b is a flopping curve. Thus $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of C_{d-1}. Suppose that $b \in C_{d-1} \setminus l$. If m'_b intersects a flopping curve, m_b is a line or a point, thus $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of C_{d-1}. In the other case, m_b is a conic. If $b \notin \cup_i \beta_i$, then $\deg C_{d-1,b} = d - 3$ by Proposition 2.1.6 (1-1). By $d \geq 6$, $C_{d-1,b}$ is not a conic, thus $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of C_{d-1}. Assume $b \in \beta_i$. Then

$\deg C_{d-1,b} = d-4$. We have only to show that if $d=6$, then $C_{d-1,b} \neq m_b$. By Proposition 2.2.2 (4), the flop is of type (a) in Proposition 2.1.6 (1-2). The line m'_b intersects three lines which are the strict transforms of three fibers of π_b contained in E'_b. On the other hand, by $E'_b \cdot C''_{d-1,b} = 2$, the curve $C''_{d-1,b}$ intersects at most two fibers of π contained in E'_b. Thus $C_{d-1,b} \neq m_b$, and $\mathcal{C} \dashrightarrow \mathcal{C}_b$ is birational at a point of C_{d-1}. $\square$

We restate the proposition in terms of the relation between C_d and multi-secant conics of C_d on B as follows:

Corollary 2.2.7. *Let b be a point of B not in any bi-secant line of C_d on B. If $d \geq 5$, then there exist finitely many k-secant conics of C_d on B through b with $k \geq 2$ if $b \notin C_d$ (resp. with $k \geq 3$ if $b \in C_d$).*

Proof. For a point $b \in B$ outside bi-secant lines of C_d on B, there exist a finite number of singular multi-secant conics of C_d through b since the number of lines through b is finite, and the number of lines intersecting both a line through b and C_d is also finite by Proposition 2.2.4 (3). Therefore we have only to consider smooth multi-secant conics q of C_d through b. By Proposition 2.1.6 (1-3), the strict transform q' of such a conic q on B'_b is a fiber of π_{2b}. If $b \notin C_d$, then q' intersects C'_b twice or more counted with multiplicities, thus by Proposition 2.2.6, the finiteness of such a q follows. We can prove the assertion in case of $b \in C_d$ similarly, thus we omit the proof. $\square$

Remark. We refine this statement in Lemmas 2.4.13 and 2.5.7.

2.3. **Curve $\mathcal{H}_1$ parameterizing marked lines.**

We fix a general $C := C_d$ as in 2.2. Let $f\colon A \to B$ be the blow-up along C. We start the study of the geometry of A. The first step consists of finding the curves, if any, which replace the lines of ordinary geometry.

2.3.1. *Construction of $\mathcal{H}_1$ and marked lines.* Set $\mathcal{H}_1 := \varphi^{-1}C \subset \mathbb{P}$ and $M := M_d$. We begin with a few corollaries of Proposition 2.2.4:

Corollary 2.3.1. *If $d \geq 2$, then $\mathcal{H}_1$ is a smooth curve of genus $d-2$ with the triple cover $\mathcal{H}_1 \to C$. In particular, if $d \geq 3$, then $\mathcal{H}_1$ is a smooth non-hyperelliptic trigonal curve of genus $d-2$.*

Proof. By Propositions 2.1.3 and 2.2.4 (1), it holds that $\mathcal{H}_1$ is smooth and the ramification for $\mathcal{H}_1 \to C$ is simple by Proposition 2.2.4 (1). Since $B_\varphi \in |-K_B|$ and $d = \deg \mathrm{C}$, we can compute $g(\mathcal{H}_1)$ by the Hurwitz formula:

$$2g(\mathcal{H}_1) - 2 = 3 \times (-2) + d \times 2, \text{ equivalently, } g(\mathcal{H}_1) = d-2.$$

$\square$

Corollary 2.3.2. *The number s of nodes of M is $\frac{(d-2)(d-3)}{2}$, whence C has $\frac{(d-2)(d-3)}{2}$ bi-secant lines on B.*

Proof. By the inductive construction of C we see that $\pi_{|\mathcal{H}_1}\colon \mathcal{H}_1 \to M$ is birational. By 2.2.4 (3) $p_a(M) = \frac{(d-1)(d-2)}{2}$. Then by 2.2.4 (3) we know the number of nodes of M since $g(\mathcal{H}_1) = d-2$. The latter half follows since a bi-secant line of C corresponds to a node of M. $\square$

Now we select some lines on B which we use in the sequel. Note that

$$\mathcal{H}_1 = \{([l], t) \mid [l] \in M, t \in C \cap l\} \subset M \times C,$$

and the elements of $\mathcal{H}_1$ deserve a name:

Definition 2.3.3. The pair of a secant line l of C on B and a point $t \in C \cap l$ is called a *marked line*.

Let (l, t) be a marked line. If $C \cap l$ is one point, then $\{t\} = C \cap l$ is uniquely determined. For a bi-secant line β_i of C, there are two choices of t. Thus $\mathcal{H}_1$ parameterizes marked lines.

2.3.2. *Lines on the blow-up A of B along C_d.*

We prove that each marked line corresponds to a curve of anticanonical degree 1 on the blow-up A of B along C. This gives us a suitable notion of line on A.

Notation 2.3.4.

(1) Let $f \colon A \to B$ be the blowing up along C and E_C the f-exceptional divisor,
(2) $\{p_{i1}, p_{i2}\} = C \cap \beta_i \subset B$,
(3) $\zeta_{ij} = f^{-1}(p_{ij}) \subset E_C \subset A$, and
(4) $\beta'_i \cap \zeta_{ij} = p'_{ij} \in E_C \subset A$,

where $i = 1, \dots, s$ and $j = 1, 2$.

Definition 2.3.5. We say that a connected curve $l \subset A$ is a *line* on A if

(i) $-K_A \cdot l = 1$, and
(ii) $E_C \cdot l = 1$.

We point out that since $-K_A = f^*(-K_B) - E_C$ and $E_C \cdot l = 1$ then $f(l)$ is a line on B *intersecting* C. More precisely:

Proposition 2.3.6. *A line l on A is one of the following curves on A:*

(i) *the strict transform of a uni-secant line of C on B, or*
(ii) *the union $l_{ij} = \beta'_i \cup \zeta_{ij}$, where $i = 1, \dots, s$ and $j = 1, 2$.*

In particular l is reduced and $p_a(l) = 0$.

Notation 2.3.7. For a line l on A, we usually denote by $\bar{l}$ its image on B.

Corollary 2.3.8. *The curve $\mathcal{H}_1 \subset \mathbb{P}$ is the Hilbert scheme of the lines of A.*

Proof. Let $\mathcal{H}'_1$ be the Hilbert scheme of lines on A, which is a locally closed subset of the Hilbert scheme of A. By the obstruction calculation of the normal bundles of the components of lines on A, it is easy to see that $\mathcal{H}'_1$ is a smooth curve. Denote by $\mathcal{U}_1 \to \mathcal{H}'_1$ the universal family of the lines on A and let $\overline{\mathcal{U}}_1$ be the image of $\mathcal{U}_1$ on $B \times \mathcal{H}'_1$ (with induced reduced structure).

Claim 2.3.9. *$\overline{\mathcal{U}} \to \mathcal{H}'_1$ is a $\mathbb{P}^1$-bundle.*

Proof of the claim. Let $\mathcal{L}$ be the pull-back of the ample generator of $\operatorname{Pic} B$ by

$$\mathcal{U}_1 \hookrightarrow A \times \mathcal{H}'_1 \to B \times \mathcal{H}'_1 \to B.$$

Since $\varrho \colon \mathcal{U}_1 \to \mathcal{H}'_1$ is flat and $h^0(l, \mathcal{L}_{|l}) = 2$ for a line l on B, $\mathcal{E} := \varrho_* \mathcal{L}$ is a locally free sheaf of rank two. $\mathbb{P}(\mathcal{E})$ is nothing but the $\mathbb{P}^1$-bundle contained in $B \times \mathcal{H}'_1$ whose fiber is the image of a line on A. This implies that $\mathbb{P}(\mathcal{E}) = \overline{\mathcal{U}}$ as schemes and $\overline{\mathcal{U}}$ is a $\mathbb{P}^1$-bundle. $\square$

By the claim same we have a natural morphism $\mathcal{H}'_1 \to \mathbb{P}^2$, whose image is M. By Proposition 2.3.6 $\mathcal{H}'_1 \to M$ is birational and surjective. Since $\mathcal{H}'_1$ and $\mathcal{H}_1$ are smooth, they are both normalizations of M, then $\mathcal{H}'_1 \simeq \mathcal{H}_1$. $\square$

Remark. For a bi-secant line β_i, we have two choices of marking, p_{i1} or p_{i2}. We describe which line on A corresponds to (β_i, p_{ij}). Denote by $\mathcal{U}_1 \to \mathcal{H}_1$ the universal family of the lines on A and consider the following diagram:

$$\begin{array}{ccc} \mathcal{U}_1 \subset & & A \times \mathcal{H}_1 \\ \downarrow & & \downarrow \\ \overline{\mathcal{U}}_1 \subset & & B \times \mathcal{H}_1. \end{array}$$

Then $\mathcal{U}_1 \to \overline{\mathcal{U}}_1$ is the blow-up along $(C \times \mathcal{H}_1) \cap \overline{\mathcal{U}}_1$, which is the union of a section of $\overline{\mathcal{U}}_1 \to \mathcal{H}_1$ consisting markings and finite set of points $(p_{i,3-j}, [\beta_i, p_{ij}])$. Thus the marked line (β_i, p_{ij}) corresponds to the line $l_{i,3-j}$.

2.4. Surface $\mathcal{H}_2$ parameterizing marked conics.

Now we define a notion of *conic* on A. We proceed as in the case of lines, first defining the notion of *marked conic*.

2.4.1. *Construction of $\mathcal{H}_2$ and marked conics.*

Definition 2.4.1. The pair of a k-secant conic q on B with $k \geq 2$ and a zero-dimensional subscheme $\eta \subset C$ of length two contained in $q_{|C}$ is called a *marked conic.*

From now on, we assume that $d \geq 3$.

Marked conics are parameterized by

$$\mathcal{H}'_2 := \{([q], [\eta]) \mid [q] \in \overline{\mathcal{H}}'_2, \eta \subset C \cap q\} \subset \overline{\mathcal{H}}'_2 \times S^2C$$

with reduced structure, where $\overline{\mathcal{H}}'_2 \subset \mathbb{P}^4$ is the locus of multi-secant conics of C on B.

By Corollary 2.1.7 and $d \neq 1$, the natural projection of $\mathcal{H}'_2 \to S^2C$ is one to one outside $[\beta_{i|C}]$ and the diagonal of S^2C, thus by the Zariski main theorem, it is an isomorphism outside $[\beta_{i|C}]$ and the diagonal of S^2C.

We denote by e'_i the fiber of $\mathcal{H}'_2 \to S^2C$ over a $[\beta_{i|C}]$. Since B is the intersection of quadrics, any conic cannot intersect a line twice properly. Thus any conic $\supset \beta_{i|C}$ contains β_i. This implies that $e'_i \simeq \mathbb{P}^1$, and e'_i parameterizes marked conics of the form

$$\{([\beta_i \cup \alpha], [\beta_{i|C}]) \mid \alpha \text{ is a line such that } \alpha \cap \beta_i \neq \emptyset\}.$$

Over the diagonal of S^2C, $\mathcal{H}'_2 \to S^2C$ is finite since for $t \in C$, there exist a finite number of reducible conics with t as a singular point or conics tangent to C at t.

Hence $\mathcal{H}'_2$ is the union of the unique two-dimensional component, which dominates S^2C, and possibly lower dimensional components mapped into the diagonal of S^2C or e'_i. Note that $\mathcal{H}'_2 \to \overline{\mathcal{H}}'_2$ is finite since choices of markings of a multi-secant conic of C is finitely many by $d \geq 3$.

Claim 2.4.2. *e'_i is contained in the unique two-dimensional component of $\mathcal{H}'_2$.*

Proof. We have only to prove that $\overline{\mathcal{H}}_2'$ is two-dimensional near the generic point of the image of e_i' since $\mathcal{H}_2' \to \overline{\mathcal{H}}_2'$ is one to one near the generic point of the image of e_i'. Let $\mathcal{V}_2 \to \mathcal{H}_2^B \simeq \mathbb{P}^4$ be the universal family of conics on B and $\overline{\mathcal{H}}_2''$ the inverse image of $C \times C$ by $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \to B \times B$. Since the morphism $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \to \mathcal{V}_2 \to \mathbb{P}^4$ is flat, $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2$ is purely six-dimensional. Thus any component of $\overline{\mathcal{H}}_2''$ has dimension greater than or equal to two. Though the inverse image of the diagonal of $C \times C$ is three-dimensional, any other component of $\overline{\mathcal{H}}_2''$ is at most two-dimensional by a similar investigation to $\mathcal{H}_2'$. Thus $\overline{\mathcal{H}}_2'$ is two-dimensional near the generic point of the image of e_i' since $\overline{\mathcal{H}}_2'$ is the image of the two-dimensional part of $\overline{\mathcal{H}}_2''$ by $\mathcal{V}_2 \times_{\mathbb{P}^4} \mathcal{V}_2 \to \mathbb{P}^4$ near the generic point of the image of e_i'. □

Notation 2.4.3. Let $\mathcal{H}_2$ be the normalization of the unique two-dimensional component of $\mathcal{H}_2'$ and $\overline{\mathcal{H}}_2 \subset \overline{\mathcal{H}}_2'$ the image of $\mathcal{H}_2$. Denote by η the natural morphism $\mathcal{H}_2 \to S^2C$. Set

$$c_i := [\beta_{i|C}] \in S^2C \simeq \mathbb{P}^2,$$

and

$$e_i := \eta^{-1}(c_i),$$

where $i = 1, \ldots, s$.

By the above consideration, $\eta\colon \mathcal{H}_2 \to S^2C$ is isomorphic outside $[\beta_{i|C}]$ and $\mathcal{H}_2 \to \overline{\mathcal{H}}_2$ is the normalization. Thus we see that $\mathcal{H}_2$ parameterizes marked conics outside the inverse image of c_i. We need to understand the inverse image by η of the diagonal.

Claim 2.4.4. *Assume that $([q], [2b]) \in \mathcal{H}_2$ for $b \in C$ and a conic q. Then*

(1) *q is reduced,*
(2) *if q is smooth at b, then q is tangent to C at b, and*
(3) *if q is singular at b, then the strict transform of q is connected on A. Moreover, $b \notin \beta_i$ nor B_φ.*

Proof. By Proposition 2.1.6 (1-3) and a degeneration argument, q corresponds to the fiber of π_{2b} through the point t' in $C_b'' \cap E_b'$ coming from $t := C_b' \cap E_b$.

(1) Assume by contradiction that q is non-reduced. By Proposition 2.1.4, q is a multiple of a special line l. By Proposition 2.2.2 (4), l is a uni-secant line of C. Let m be the other line through b (by generality of C, we have $l \neq m$). Let l' and m' be the strict transforms of l and m on B_b respectively. By Proposition 2.1.6 (1-3), the fiber of π_{2b} through t' is the strict transform of the line in E_b joining $l' \cap E_b$ and $m' \cap E_b$. Hence by the assumption, $l' \cap E_b$, $m' \cap E_b$ and $C_b' \cap E_b$ are collinear. By dimension count similar to the proof of Proposition 2.2.2, we can prove that a general C does not satisfy this condition.

(2) This follows from the previous discussion.

(3) Set $q = l_1 \cup l_2$, where l_1 and l_2 are the irreducible components of q, and let l_i' be the strict transform of l_i on B_b. By (1), it holds $l_1 \neq l_2$. Then the fiber of π_{2b} corresponding to q is the strict transform of the line on E_b through $E_b \cap l_1'$ and $E_b \cap l_2'$. Note that A is obtained from B_b by blow- up B_b along C_b' and then contracting the strict transform of E_b. Thus the former half of the assertion follows. The latter half follows again by simple dimension count. □

2.4.2. *Conics on A.*

Definition 2.4.5. We say that a curve $q \subset A$ is a *conic* on A if

(i) q is connected and reduced, (ii) $-K_A \cdot q = 2$,
(iii) $E_C \cdot q = 2$, and (iv) $p_a(q) = 0$.

Using this definition, we can classify conics on A similarly to Proposition 2.3.6:

Proposition 2.4.6. *Let q be a conic on A. Then $\overline{q} := f(q) \subset B$ is a k-secant conic of C with $k \geq 2$. Moreover one of the following holds*:

(a) *$\overline{q}$ is smooth at $\overline{q} \cap C$. q is the union of the strict transform q' of $\overline{q}$ and $k-2$ distinct fibers $\zeta_1, \dots, \zeta_{k-2}$ of E_C such that $\zeta_i \cap q' \neq \emptyset$,*
(b) *$\overline{q}$ is the union of two uni-secant lines $\overline{l}$ and $\overline{m}$ such that $C \cap \overline{l} \cap \overline{m} \neq \emptyset$. q is the union of the strict transforms l and m of $\overline{l}$ and $\overline{m}$ respectively* (*we assume that $l \cap m \neq \emptyset$*), *or*
(c) *$\overline{q}$ is the union of β_i and a line $\overline{r}$ through a p_{ij}. q is the union of the fiber ζ_{ij} over p_{ij} and the strict transforms β_i' and r' of β_i and $\overline{r}$ respectively.*

Notation 2.4.7. We usually denote by $\overline{q} \subset B$ the image of a conic q on A.

Let $\mathcal{H}_2^A$ be the normalization of the two-dimensional part of the Hilbert scheme of conics on A, which is a locally closed subset of the Hilbert scheme of A. Let $\mu\colon \mathcal{U}_2 \to \mathcal{H}_2^A$ be the pull-back of the universal family of conics on A.

Lemma 2.4.8. *Let $\overline{\mathcal{U}}_2$ be the image of $\mathcal{U}_2$ on $B \times \mathcal{H}_2^A$* (*with induced reduced structure*) *then $\overline{\mathcal{U}}_2 \to \mathcal{H}_2^A$ is a conic bundle.*

Proof. The proof is similar to that of Claim 2.3.9.

Let $\mathcal{L}$ be the pull-back of the ample generator of $\operatorname{Pic} B$ by

$$\mathcal{U}_2 \hookrightarrow A \times \mathcal{H}_2^A \to B \times \mathcal{H}_2^A \to B.$$

Since $\mu\colon \mathcal{U}_2 \to \mathcal{H}_2^A$ is flat and $h^0(q, \mathcal{L}_{|q}) = 3$ for a conic q on A (recall that q is reduced), then $\mathcal{E} := \mu_* \mathcal{L}$ is a locally free sheaf of rank 3. Letting $\mathbb{P}^6 = \langle B \rangle$, $\mathbb{P}(\mathcal{E})$ is the $\mathbb{P}^2$-bundle contained in $\mathbb{P}^6 \times \mathcal{H}_2^A$ whose fiber is the plane spanned by the image of a conic on A. Let $\mathcal{Q} := (B \times \mathcal{H}_2^A) \cap \mathbb{P}(\mathcal{E})$, where the intersection is taken in $\mathbb{P}^6 \times \mathcal{H}_2^A$. A scheme theoretic fiber of $\mathcal{Q} \to \mathcal{H}_2^A$ is the image of a conic of A since B is the intersection of quadrics. Then $\mathcal{Q} = \overline{\mathcal{U}}_2$ as schemes and $\overline{\mathcal{U}}_2$ is a conic bundle. $\square$

Proposition 2.4.9. *The two surfaces $\mathcal{H}_2^A$ and $\mathcal{H}_2$ are isomorphic.*

Proof. By Lemma 2.4.8, there exists a natural morphism $\overline{\nu}\colon \mathcal{H}_2^A \to \overline{\mathcal{H}}_2'$. By Proposition 2.4.6, $\overline{\nu}$ is finite and birational, hence $\overline{\nu}$ lifts to the morphism $\nu\colon \mathcal{H}_2^A \to \mathcal{H}_2$ since $\mathcal{H}_2 \to \overline{\mathcal{H}}_2$ is the normalization. By the Zariski main theorem, ν is an inclusion. By Claim 2.4.4 (1) and (3), and Proposition 2.4.6, ν is also surjective. $\square$

By Proposition 2.4.9 we can pass freely from conics on A, that is elements of $\mathcal{H}_2^A$ to marked conics and vice-versa according to the kind of argument we will need. In particular we can speak of the universal family $\mu\colon \mathcal{U}_2 \to \mathcal{H}_2$ of marked conics meaning $\mathcal{U}_2 := \mathcal{U}_2^A$ and $\mathcal{H}_2^A$ identified to $\mathcal{H}_2$ via ν.

Corollary 2.4.10. *The Hilbert scheme of conics on A is an irreducible surface (and $\mathcal{H}_2$ is the normalization). The normalization is injective, namely, $\mathcal{H}_2$ parameterizes conics on A in one to one way.*

Proof. By Proposition 2.4.6, the image of $\mathcal{H}_2$ in the Hilbert scheme parameterizes all the conics, thus the first part follows.

For the second part, we have already seen that $\mathcal{H}_2$ parameterizes conics on A in one to one way outside $\cup_i e_i$. Let α be a general line intersecting β_i, and α' the strict transform of α on A. By easy obstruction calculation, we see that the Hilbert scheme of conics on A is smooth at $[\beta'_i \cup \alpha']$. Thus general points of e_i also parameterizes conics on A. Then, however, since $e'_i \simeq \mathbb{P}^1$, where e'_i is the inverse image of $[\beta_{i|C}]$ by $\mathcal{H}'_2 \to S^2C$, it holds that $e_i \simeq e'_i \simeq \mathbb{P}^1$ ($\mathcal{H}_2 \to S^2C$ has only connected fibers). This implies the assertion. □

2.4.3. *Description of $\mathcal{H}_2$.*

We want to investigate further the morphism $\eta\colon \mathcal{H}_2 \to S^2C \simeq \mathbb{P}^2$.

Notation 2.4.11. For a point $b \in C$, set

$$L_b := \overline{\{[q] \in \mathcal{H}_2 \mid \exists, b' \neq b, f(q) \cap C = \{b, b'\}\}}.$$

By Corollary 2.1.7, $\eta(L_b)$ is a line in $S^2C \simeq \mathbb{P}^2$.

To understand better $\eta\colon \mathcal{H}_2 \to \mathbb{P}^2$ we need to find special loci inside $\mathcal{H}_2$. A natural step is to study the locus of conics which intersect a fixed line.

Let $\mathcal{U}'_1 \subset \mathcal{U}_2 \times \mathcal{H}_1$ be the pull-back of $\mathcal{U}_1$ via the following diagram:

$$\begin{tikzcd} \mathcal{U}'_1 \subset \mathcal{U}_2 \times \mathcal{H}_1 \arrow[r] \arrow[d] & A \times \mathcal{H}_1 \supset \mathcal{U}_1 \arrow[d] \\ \widehat{\mathcal{D}}_1 \subset \mathcal{H}_2 \times \mathcal{H}_1 \arrow[r] & \mathcal{H}_1, \end{tikzcd} \tag{2.6}$$

where $\widehat{\mathcal{D}}_1$ is the image of $\mathcal{U}'_1$ on $\mathcal{H}_2 \times \mathcal{H}_1$.

By definition

$$\widehat{\mathcal{D}}_1 = \{([q], [l]) \mid q \cap l \neq \emptyset\} \subset \mathcal{H}_2 \times \mathcal{H}_1.$$

First we need to know which component of $\widehat{\mathcal{D}}_1$ is divisorial or dominates $\mathcal{H}_1$.

Let $\psi\colon \mathcal{U}_2 \to A$ be the morphism obtained via the universal family $\mu\colon \mathcal{U}_2 \to \mathcal{H}_2$. Next lemma is necessary to prove the finitess of ψ outside $\cup_{i=1}^s \beta'_i \subset A$.

Lemma 2.4.12. *Let $\overline{l}$ be a general uni-secant line of C and $l_b \subset \mathbb{P}^2$ the image of $\overline{l}$ by the double projection from a point b. For a general point $b \notin C$, $\deg C_b = d$ and $C_b \cup l_b$ has only simple nodes. For a general point b of C, $\deg C_b = d - 2$ and $C_b \cup l_b$ has only simple nodes.*

Proof. The claims for $\deg C_b$ follows from Propositions 2.1.6 (1-1) and 2.2.6. As for the singularity of $C_b \cup l_b$, the claim follows from simple dimension count. For simplicity, we only prove that for a general point b, C_b has only simple nodes. By Proposition 2.2.2, we may assume that any multi-secant conic through b is smooth, bi-secant and intersects C simply. Let $\overline{q}$ be a smooth bi-secant conic through b. We may assume that $\mathcal{N}_{\overline{q}/B} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$. Let q' be the strict transform of $\overline{q}$ on B'_b. Let $\widetilde{B}' \to B'_b$ be the blow-up along q', $E_{q'}$ the exceptional divisor and $\widetilde{C}''$ the strict transform of C''_b. Note that $E_{q'} \simeq \mathbb{P}^1 \times \mathbb{P}^1$ since $\mathcal{N}_{q'/B'_b} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Then C_b has simple

nodes at the image of q' if and only if the two points in $E_{q'} \cap \widetilde{C}''$ does not belong to the same ruling with the opposite direction to a fiber of $E_{q'} \to q'$. Let $\widetilde{B}_{\overline{q}} \to B$ be the blow-up along $\overline{q}$, $E_{\overline{q}}$ the exceptional divisor and $\widetilde{C}$ the strict transform of C. It is easy to see that a ruling of $E_{\overline{q}}$ with the opposite direction to a fiber of $E_{\overline{q}} \to \overline{q}$ corresponds to that of $E_{q'}$ with the opposite direction to a fiber of $E_{q'} \to q'$. Thus C_b has simple nodes at the image of q' if and only if the two points in $E_{\overline{q}} \cap \widetilde{C}$ does not belong to the same ruling with the opposite direction to a fiber of $E_{\overline{q}} \to \overline{q}$. We can show that this is the case for a general b by simple dimension count. $\square$

From now on, we assume $d \geq 5$ throughout the paper since we need Proposition 2.2.6.

We do not have the finiteness of ψ all over A. To obtain a finite morphism, we blow-up A more in 2.5.1. Till now we can prove:

Proposition 2.4.13. *ψ is finite of degree $n := \frac{(d-1)(d-2)}{2}$ and flat outside $\cup_{i=1}^{s}\beta'_i$.*

Proof. Let $a \in A \setminus \cup_{i=1}^{s}\beta'_i$ and set $b := f(a)$. If $b \notin C$, then the finiteness of ψ over a follows from Corollary 2.2.7. Moreover, by Lemma 2.4.12, the number of conics through a general a is n. Thus $\deg \psi = n$. We will prove that ψ is finite over $a \in E_C \setminus \cup_{i=1}^{s}\beta'_i$. Once we prove this, the assertion follows. Indeed, $\mathcal{U}_2$ is Cohen-Macaulay since $\mathcal{H}_2$ is smooth and any fiber of $\mathcal{U}_2 \to \mathcal{H}_2$ is reduced, thus ψ is flat.

Let $a \in E_C \setminus \cup_{i=1}^{s}\beta'_i$. The assertion is equivalent to that only finitely many conics belonging to L_b pass through a. If $b \notin \cup_{i=1}^{s}\beta_i$, then L_b is irreducible. If $b \in \cup_{i=1}^{s}\beta_i$, then $L_b = L'_b + e_i$, where L'_b is the strict transform of $\eta(L_b)$ whence is irreducible. Note that almost all the conics belonging to e_i does not pass through $a \notin \cup_{i=1}^{s}\beta'_i$. Let $S_b \subset A$ be the locus swept by the conics of the family L_b if $b \notin \cup_{i=1}^{s}\beta_i$, or the locus swept by the conics of the family L'_b if $b \in \cup_{i=1}^{s}\beta_i$. S_b is irreducible. Let $\overline{S}_b := f(S_b)$, $\overline{S}'_b$ and $\overline{S}''_b$ the strict transforms of $\overline{S}_b$ on B_b and B'_b respectively. Then $\overline{S}''_b = \pi_{2b}^{*}C_b$. Let $d_b := \deg C_b$. By Proposition 2.1.6 (1-1), $d_b = d-2$ if $b \notin \cup_{i=1}^{s}\beta_i$, or $d-3$ if $b \in \cup_{i=1}^{s}\beta_i$. Since $\overline{S}''_b \sim d_b L$ and $L = H - 2E'_b$, we have $\overline{S}'_{b|E_b}$ is a curve of degree $2d_b$ in $E_b \simeq \mathbb{P}^2$.

Since A is obtained from B_b by blowing up C'_b and then contracting the strict transform of E_b, a point a over b corresponds to a line l_a in E_b through $t := E_b \cap C'_b$. If C''_b does not intersect fibers of π_{2b} contained in E'_b, then $\overline{S}'_{b|E_b}$ is irreducible. Thus no l_a is contained in $\overline{S}'_{b|E_b}$ and we are done. Assume that C''_b intersects a fiber l' of π_{2b} contained in E'_b. By Claim 2.4.4 (3), $b \notin B_\varphi$ nor $b \notin \cup_{i=1}^{s}\beta_i$ for a general C. Since L_b is irreducible, it suffices to prove the finiteness and nonemptyness of the set of conics through a general point a over b. Equivalently, we have only to show that a general l_a intersects $\overline{S}'_{b|E_b}$ outside t. Since l' intersects C''_b simply at one point, C_b is smooth at the image t' of l' on $\mathbb{P}^2$. Thus $\overline{S}'_{b|E_b} = C'''_b + l$, where C'''_b and l are the strict transforms of C_b and l'. Note that C'''_b is smooth at t and $\deg C'''_b = 2d_b - 1 = 2d - 5 \geq 5$ by $d \geq 5$. Thus a general l_a intersect C'''_b outside t. $\square$

Remark. Though we do not need it later, we describe the fiber of ψ over a general point $a \in E_C \setminus \cup_{i=1}^{s}\beta'_i$ for reader's convenience.

Set $b := f(a)$. As in the proof of Proposition 2.4.13, a point a over b corresponds to a line l_a in E_b passing through $E_b \cap C'_b$. By Lemma 2.4.12, it holds that $\deg C_b =$

$d-2$ and C_b has $\frac{(d-3)(d-4)}{2}$ simple nodes for a general $b \in C$. This means that $\frac{(d-3)(d-4)}{2}$ tri-secant conics pass through b. By Proposition 2.4.6, corresponding to a tri-secant conic $\overline{q}$, there is a unique conic q on A containing the fiber of E_C over b and such a conic on A contains a. Thus we obtain $\frac{(d-3)(d-4)}{2}$ conics through a. By definition of L_b, these conics do not belong to L_b.

We need more $n - \frac{(d-3)(d-4)}{2} = 2d-5$ conics through a. We show that there exist $2(d-2)-1$ conics through a on A coming from the family parameterized by L_b. We use the notation of the proof of Proposition 2.4.13. For a general $b \in C$, C''_b does not intersect fibers of π_{2b} contained in E'_b. Thus $\overline{S}'_{b|E_b}$ is an irreducible curve of degree $2(d-2)$ on E_b. Thus there are $2(d-2)$ intersection points of $\overline{S}'_{b|E_b}$ and l_a. Among those, the intersection point $C'_b \cap E_b$ does not correspond to a conic on A through a since it comes from the tangent of C. Thus we have $2(d-2)-1$ conics as desired.

We need to study mutual intersection of a conic and a line in special cases. Let $\mathcal{F} \subset \mathcal{H}_2 \times \mathcal{H}_1$ be the image in $\mathcal{H}_2 \times \mathcal{H}_1$ of the inverse image of $((\cup \beta'_i) \times \mathcal{H}_1) \cap \mathcal{U}_1$; that is

$$\mathcal{F} := \{([q],[l]) \mid q \cap \beta'_i \cap l \neq \emptyset\}.$$

A point $([q],[l]) \in \mathcal{F}$ iff i) $l = l_{ij}$ and $q \cap \beta'_i \neq \emptyset$ or ii) $l \neq l_{ij}$, $l \cap \beta'_i \neq \emptyset$ and $q \cap \beta'_i \cap l \neq \emptyset$. For every $i = 1, \ldots, s$, $j = 1, 2$ the family of those $([q],[l])$ which satisfies i) or ii) has dimension one and clearly does not dominate $\mathcal{H}_1$.

Corollary 2.4.14. *Any component of $\widehat{\mathcal{D}}_1$ which is not contained in $\mathcal{F}$ dominates $\mathcal{H}_1$. Moreover, any non-divisorial component of $\widehat{\mathcal{D}}_1$ outside $\mathcal{F}$ (if it exists) is a one-dimensional component whose generic point parameterizes reducible conics, namely, a one-dimensional component of*

$$\{([q],[l]) \mid l \subset q\}.$$

Remark. Here we leave the possibility that a one-dimensional component whose generic point parameterizes reducible conics is contained in a divisorial component of $\widehat{\mathcal{D}}_1$. We, however, prove that this is not the case in Corollary 2.4.19. Hence, finally, the fiber of $\widehat{\mathcal{D}}_1 \to \mathcal{H}_1$ over a general $[l] \in \mathcal{H}_1$ parameterizes conics which properly intersect l.

Proof. By Proposition 2.4.13, $\mathcal{U}_2 \to A$ is finite and flat outside $\cup \beta'_i$. Then $\mathcal{U}_2 \times \mathcal{H}_1 \to A \times \mathcal{H}_1$ is flat outside $(\cup \beta'_i) \times \mathcal{H}_1$. By base change, $\mathcal{U}'_1 \to \mathcal{U}_1$ is flat and finite outside $((\cup \beta'_i) \times \mathcal{H}_1) \cap \mathcal{U}_1$. Then every irreducible component of $\mathcal{U}'_1$ which is not mapped to $((\cup \beta'_i) \times \mathcal{H}_1) \cap \mathcal{U}_1$ is two-dimensional, and dominates $\mathcal{U}_1$, hence dominates $\mathcal{H}_1$. Therefore any component of $\widehat{\mathcal{D}}_1$ which is not contained in $\mathcal{F}$ dominates $\mathcal{H}_1$.

We find a possible non-divisorial component of $\widehat{\mathcal{D}}_1$ outside $\mathcal{F}$. Let $\gamma \subset \mathcal{U}'_1$ be a curve mapped to a point, say, $([q],[l])$ on $\mathcal{H}_2 \times \mathcal{H}_1$. The image of γ on A is an irreducible component of q, say, q_1. The image of γ on $\mathcal{U}_1$ is $q_1 \times [l]$, thus q_1 is also an irreducible component of l. We have the following three possibilities:

(1) l is irreducible, hence $q_1 = l$ and $q = l \cup m$, where m is another line. Such $([q],[l])$ form the one-dimensional family of reducible conics,

(2) $l = l_{ij}$ and $\beta'_i \subset q$. Namely $[q] \in e_i$, or $q = \beta'_i \cup \alpha \cup \zeta_{ik}$, where α is the strict transform of a line on B intersecting β_i and C outside $\beta_i \cap C$, or

(3) $l = l_{ij}$ and $\zeta_{ij} \subset q$ and $f(q)$ is a tri- or quadri-secant conic of C such that $p_{ij} \in f(q)$.

Thus we have the second assertion. $\square$

Notation 2.4.15. Let $\mathcal{D}_1 \subset \mathcal{H}_2 \times \mathcal{H}_1$ be the divisorial part of $\widehat{\mathcal{D}}_1$. Since $\mathcal{H}_1$ is a smooth curve $\mathcal{D}_1 \to \mathcal{H}_1$ is flat. Let D_l be the fiber of $\mathcal{D}_1 \to \mathcal{H}_1$ over $[l] \in \mathcal{H}_1$. Clearly we can write $D_l \hookrightarrow \mathcal{H}_2$.

Next two lemmas are basic to understand the geometry of $\mathcal{H}_2$.

Lemma 2.4.16. *Let $\overline{l}_1$ and $\overline{l}_2$ be two general secant lines of C such that $\overline{l}_1 \cap \overline{l}_2 = \emptyset$. Let $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ be the successive linear projections from $\overline{l}_1$ and then the strict transform of $\overline{l}_2$ on Q. Let $\overline{l}$ be another general secant line of C, and C' and $\overline{l}' \subset \mathbb{P}^2$ be the images of C and $\overline{l}$ respectively. Then $C \cup \overline{l} \dashrightarrow C' \cup \overline{l}'$ is birational and $C' \cup \overline{l}'$ has only simple nodes as its singularities, where, by birational, we means that $\deg C' \cup \overline{l}' = d-1$. In particular (since $\deg C' = d-2$ and C' is rational) C' has $\frac{(d-3)(d-4)}{2}$ simple nodes, equivalently, there exist $\frac{(d-3)(d-4)}{2}$ bi-secant conics of C intersecting both $\overline{l}_1$ and $\overline{l}_2$.*

Proof. We show the assertion using the inductive construction of $C = C_d$. The assertion follows for $d = 3$ directly. Consider a smoothing from $C_{d-1} \cup \overline{m}$ to C_d. Let $\overline{m}_1$ and $\overline{m}_2$ two general secant lines of C_{d-1} such that $\overline{m}_1 \cap \overline{m}_2 = \emptyset$. Let $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ be the successive linear projections from $\overline{m}_1$ and then from the strict transform of $\overline{m}_2$ on Q. Let $\overline{r}$ be another general secant line of C_{d-1}, and $C'_{d-1}, \overline{m}'$ and $\overline{r}' \subset \mathbb{P}^2$ be the images of C_{d-1}, $\overline{m}$ and $\overline{r}$ respectively. Then we have only to show that $C_{d-1} \cup \overline{m} \cup \overline{r} \dashrightarrow C'_{d-1} \cup \overline{m}' \cup \overline{r}'$ is birational and $C'_{d-1} \cup \overline{m}' \cup \overline{r}'$ has only simple nodes as its singularities assuming $C_{d-1} \cup \overline{r} \dashrightarrow C'_{d-1} \cup \overline{r}'$ is birational and $C'_{d-1} \cup \overline{r}'$ has only simple nodes as its singularities.

Since $\overline{m}$ is also general, $C_{d-1} \cup \overline{m} \dashrightarrow C'_{d-1} \cup \overline{m}'$ is birational and $C'_{d-1} \cup \overline{m}'$ has only simple nodes as its singularities. Thus $C_{d-1} \cup \overline{m} \cup \overline{r} \dashrightarrow C'_{d-1} \cup \overline{m}' \cup \overline{r}'$ is clearly birational. To show $C'_{d-1} \cup \overline{m}' \cup \overline{r}'$ has only simple nodes as its singularities, it suffices to prove that there are no secant conics of C_{d-1} intersecting all the $\overline{m}_1$, $\overline{m}_2$, $\overline{m}$ and $\overline{r}$. This follows from the fact that a secant conic $\overline{q}$ of C_{d-1} intersects finitely many secant lines of C_{d-1} by $M(\overline{q}) \not\subset M(C_{d-1})$. $\square$

Lemma 2.4.17. *Let $\overline{l}_0$ be a general secant line of C. Let $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ be the successive linear projections from $\overline{l}_0$ and then the strict transform of a β_i on Q. Let $\overline{l}$ be another general secant line of C, and C' and $\overline{l}' \subset \mathbb{P}^2$ be the images of C and $\overline{l}$ respectively. Then $C \cup \overline{l} \dashrightarrow C' \cup \overline{l}'$ is birational and $C' \cup \overline{l}'$ has only simple nodes as its singularities. In particular (since $\deg C' = d-3$ and C' is rational) C' has $\frac{(d-4)(d-5)}{2}$ simple nodes, equivalently, there exist $\frac{(d-4)(d-5)}{2}$ bi-secant conics of C intersecting β_i and $\overline{l}_0$ except conics containing β_i.*

Proof. Similarly to the previous lemma, we show the assertion using the inductive construction of $C = C_d$. The assertion follows for $d = 4$ directly. Consider a smoothing from $C_{d-1} \cup \overline{m}$ to C_d. Let $\overline{m}_0$ be a general secant line of C_{d-1}, and β a bi-secant line of $C_{d-1} \cup \overline{m}$ different from two lines through $C_{d-1} \cap \overline{m}$. Let $B \dashrightarrow Q \dashrightarrow \mathbb{P}^2$ be the successive linear projections from $\overline{m}_0$ and then the strict transform of β on Q. Let $\overline{r}$ be another general secant line of C_{d-1}, and $C'_{d-1}, \overline{m}'$ and $\overline{r}' \subset \mathbb{P}^2$ be the images of C_{d-1}, $\overline{m}$ and $\overline{r}$ respectively.

First we suppose that β is a bi-secant line of C_{d-1}. Then we have only to show that $C_{d-1} \cup \overline{m} \cup \overline{r} \dashrightarrow C'_{d-1} \cup \overline{m}' \cup \overline{r}'$ is birational and $C'_{d-1} \cup \overline{m}' \cup \overline{r}'$ has only

simple nodes as its singularities assuming $C_{d-1} \cup \overline{r} \dashrightarrow C'_{d-1} \cup \overline{r}'$ is birational and $C'_{d-1} \cup \overline{r}'$ has only simple nodes as its singularities. The proof is the same as that of Lemma 2.4.16, so we omit it.

Next suppose that β is a uni-secant line of C_{d-1} intersecting $\overline{m}$ outside $C_{d-1} \cap \overline{m}$. Note that, by the projection $B \dashrightarrow \mathbb{P}^2$, $\overline{m}$ is contracted to a point. Moreover, β is a general uni-secant line since so is $\overline{m}$. Thus, by Lemma 2.4.16, $C_{d-1} \cup \overline{m} \cup \overline{r} \dashrightarrow C'_{d-1} \cup \overline{m}' \cup \overline{r}'$ is birational and $C'_{d-1} \cup \overline{m}' \cup \overline{r}'$ has only simple nodes as its singularities. $\square$

Now we reach the precise description of $\mathcal{H}_2$.

Theorem 2.4.18. *$\eta\colon \mathcal{H}_2 \to \mathbb{P}^2$ is the blow-up at $c_1, \dots, c_s$ and e_i are η-exceptional curves. Moreover $\mathcal{H}_2$ has the following properties:*

(1)
$$D_l \sim (d-3)h - \sum_{i=1}^{s} e_i,$$
where h is the strict transform of a general line on $\mathbb{P}^2$.

(2)
$$h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}((d-4)h - \sum_{i=1}^{s} e_i)) = 0.$$

(3) *$|D_l|$ is base point free. In case of $d=5$, the image of $\Phi_{|D_l|}$ is $\check{\mathbb{P}}^2$. In case of $d \geq 6$, D_l is very ample and $|D_l|$ embeds $\mathcal{H}_2$ into $\check{\mathbb{P}}^{d-3}$.*

(4) *If $d \geq 6$, then $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$ is projectively Cohen-Macaulay and is the intersection of cubics .*

Remark. (1) If $d \geq 6$, then $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$ is so called the *White surface* (see [Whi24] and [Gim89]). In [Man01], the white surface attains the maximal degree among projectively Cohen Macaulay rational surfaces in a fixed projective space.

(2) We use the dual notation $\check{\mathbb{P}}^{d-3}$ for later convenience.

Proof. (1) Let $\pi_C\colon C \times C \to S^2C$ be the natural projection and L'_b a ruling of $C \times C$ in one fixed direction such that $\pi_C(L'_b) = \eta(L_b)$. By applying the Bertini theorem to $|L'_b|$, we see that $\pi_C^*\eta(D_l)$ and L'_b intersect simply for a general $b \in C$ whence $\eta(D_l)$ intersects $\eta(L_b)$ simply since π_C is étale at $\pi_C^*\eta(D_l) \cap L'_b$. Then D_l intersects L_b simply since η is isomorphic at $D_l \cap L_b$. For a general $b \in C$, we consider the double projection $\pi_b\colon B \dashrightarrow \mathbb{P}^2$ from b as in Proposition 2.1.6 (1). Let $\overline{l}$ be a general line on B intersecting C and l_b the image of $\overline{l}$ by π_b. We can assume that $b \neq c := C \cap \overline{l}$. Obviously l_b is a line. By Lemma 2.4.12, $\deg C_b = d-2$ and the curve $C_b \cup l_b$ has only simple nodes. Hence the number of points in $C_b \cap l_b$ is $d-2$, one of them counts for the unique conic through b and c. This last conic gives a conic on A which does not intersects l. The other $d-3$ points count for elements of D_l. Then $\eta(D_l)$ is a curve of degree $d-3$.

Let $\overline{l}_1$ and $\overline{l}_2$ be two general secant lines of C such that $\overline{l}_1 \cap \overline{l}_2 = \emptyset$. By Lemma 2.4.16, $\#(D_{l_1} \cap D_{l_2}) = \frac{(d-3)(d-4)}{2}$. This immediately gives for the intersection product $D_{l_1} \cdot D_{l_2} \geq \frac{(d-3)(d-4)}{2}$. On the other hand, $D_l \cap e_i \neq \emptyset$ for a general l since $D_l \cap e_i$ contains the point corresponding to a marked conic $(\beta_i \cup \alpha, \beta_{i|C})$, where α is the unique line intersecting β_i and l. Moreover, for two general l_1 and l_2, $D_{l_1} \cap D_{l_2} \cap e_i = \emptyset$. Now the curves e_i have negative self intersection then

$D_{l_1}\cdot D_{l_2}\le (d-3)^2-s=\frac{(d-3)(d-4)}{2}$. Therefore $D_{l_1}\cdot D_{l_2}=\frac{(d-3)(d-4)}{2}$. Moreover $e_i^2=-1$ and since $e_i\cap e_j=\emptyset$ we obtain that $\eta\colon \mathcal{H}_2\to\mathbb{P}^2$ is the blow-up at $c_1,\dots,c_s$. Consequently, $D_l\sim(d-3)h-\sum_{i=1}^s e_i$ for a general $[l]\in\mathcal{H}_1$, and, by the flatness of $\mathcal{D}_1\to\mathcal{H}_1$, that holds for any $[l]\in\mathcal{H}_1$.

(2) Let $L'_{p_{ij}}=L_{p_{ij}}-e_i$ (note that $e_i\subset L_{p_{ij}}$). We see that $L'_{p_{ij}}\subset D_{l_{ij}}$ and $D_{l_{i1}}-L'_{p_{i1}}=D_{l_{i2}}-L'_{p_{i2}}$, which we denote by D_{β_i}. Note that

$$D_{\beta_i}\sim(d-4)h-\sum_{k\ne i}e_k.$$

It is easy to see that D_{β_i} have the following properties:

$$D_{\beta_i}\cap e_i=\emptyset. \tag{2.7}$$

$$D_{\beta_i}\cap D_{\beta_j}\cap D_{\beta_k}=\emptyset. \tag{2.8}$$

We only prove (2.7). Since $D_{\beta_i}\cap e_i\ne\emptyset$ would imply that e_i is a component of D_{β_i}, it suffices to prove that, for a general l, $D_{\beta_i}\cap D_l$ does not contain a point of e_i. By Lemma 2.4.17, $D_{\beta_i}\cap D_l$ contains $\frac{(d-4)(d-5)}{2}$ points corresponding to bi-secant conics intersecting β_i and l except conics containing β_i. On the other hand, we have $D_l\cdot D_{\beta_i}=\frac{(d-4)(d-5)}{2}$, thus the conics we count in Lemma 2.4.17 correspond to all the intersection of $D_{\beta_i}\cap D_l$. Consequently, $D_{\beta_i}\cap D_l$ does not contain a point of e_i.

By (2.7) and the trivial equality

$$(d-4)h-\sum_{i\ge k+1}e_i=D_{\beta_k}+e_1+\cdots+e_{k-1},$$

we obtain $e_k\not\subset \operatorname{Bs}|(d-4)h-\sum_{i\ge k+1}e_i|$.

Since $\mathcal{O}_{\mathcal{H}_2}((d-4)h-\sum_{i\ge k+1}e_i)\otimes_{\mathcal{O}_{\mathcal{H}_2}}\mathcal{O}_{e_k}\simeq\mathcal{O}_{e_k}$ we have that

$$H^0(\mathcal{H}_2,\mathcal{O}_{\mathcal{H}_2}((d-4)h-\sum_{i\ge k+1}e_i))\to H^0(\mathcal{H}_2,\mathcal{O}_{e_k})$$

is surjective. Hence by the exact sequence

$$0\to\mathcal{O}_{\mathcal{H}_2}((d-4)h-\sum_{i\ge k}e_i)\to\mathcal{O}_{\mathcal{H}_2}((d-4)h-\sum_{i\ge k+1}e_i)\to\mathcal{O}_{e_k}\to 0,$$

we have $H^1(\mathcal{H}_2,\mathcal{O}_{\mathcal{H}_2}((d-4)h-\sum_{i=1}^s e_i))\simeq H^1(\mathcal{H}_2,\mathcal{O}_{\mathcal{H}_2}(d-4)h)$. Since it is easy to see that $h^1(\mathcal{H}_2,\mathcal{O}_{\mathcal{H}_2}(d-4)h)=0$, we have (2).

(3) Since no conic on A intersects all the line on A, $|D_l|$ has no base point. In case $d=5$, the image of $\Phi_{|D_l|}$ is $\mathbb{P}^2$ by $(D_l)^2=1$.

Assuming $d\ge 6$, we prove that D_l is very ample. By (2) and [DG88, Theorem 3.1], it suffices to prove that

$$h^0(\mathcal{H}_2,\mathcal{O}_{\mathcal{H}_2}(h-\sum_{j=1}^{d-3}e_{i_j}))=0$$

for any set of $d-3$ exceptional curves $e_{i_1},\dots,e_{i_{d-3}}$. Assume by contradiction that there exists an effective divisor $L\in|h-\sum_{j=1}^{d-3}e_{i_j}|$ for a set of $d-3$ exceptional curves $e_{i_1},\dots,e_{i_{d-3}}$. By $\frac{(d-2)(d-3)}{2}-(d-3)\ge 3$, we find at least three e_i such that $i\notin\{j_1,\dots,j_{d-3}\}$. For an $i\notin\{j_1,\dots,j_{d-3}\}$, noting $D_l\sim D_{\beta_i}+h-e_i$, $D_l\cdot L=0$, and $L\cdot(h-e_i)>0$, we have $L\subset D_{\beta_i}$. This contradicts (2.8) since the number of i such that $i\notin\{j_1,\dots,j_{d-3}\}$ is at least 3.

We show that $h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = d-2$. By the Riemann-Roch theorem, $\chi(\mathcal{O}_{\mathcal{H}_2}(D_l)) = d-2$. Since $h^2(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(-D_l + K_{\mathcal{H}_2})) = 0$, we see that $h^0(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = d-2$ is equivalent to $h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = 0$. Since $|D_l|$ has no base point, so is $|(d-3)h - \sum_{i\geq k+1} e_i|$. Thus the proof that $h^1(\mathcal{H}_2, \mathcal{O}_{\mathcal{H}_2}(D_l)) = 0$ is almost the same as the above one showing (2) and we omit it.

(4) follows from [Gim89, Proposition 1.1]. □

Remark. In case of $d=5$, the morphism defined by $|D_l|$ contracts three curves D_{e_i} $(i=1,2,3)$, which are nothing but the strict transforms of three lines passing through two of c_j. Namely, the composite $S^2C \leftarrow \mathcal{H}_2 \to \check{\mathbb{P}}^2$ is the Cremona transformation.

The following corollary contains the nontrivial result that for a general $[l] \in \mathcal{H}_1$, D_l parameterizes conics which properly intersect l.

Corollary 2.4.19. *For a general $[l] \in \mathcal{H}_1$, D_l does not contain any point corresponding to the line pairs $l \cup m$ with $[m] \in \mathcal{H}_1$.*

Proof. Fix $[m] \in \mathcal{H}_1$ such that $l \cup m$ is a line pair. If $(\overline{m}, b)$ is the marked line given by m then we have $d-2$ line pairs $l \cup m$, $l_1 \cup m, \dots, l_{d-3} \cup m$. Since $L_b \sim h$ then $h \cdot D_l = d-3$ and definitely $[l_1 \cup m], \dots, [l_{d-3} \cup m] \in D_l$. Thus $[l \cup m] \notin D_l$. □

2.5. **Varieties of power sums for special non-degenerate quartics F_4.**

In Proposition 2.4.13 we have seen that $\psi\colon \mathcal{U}_2 \to A$ is finite and flat outside $\cup_{i=1}^n \beta_i'$. We can modify the morphism $\psi\colon \mathcal{U}_2 \to A$ to obtain a finite one. See Proposition 2.5.7, which is the goal of 2.5.1. This and our understanding of the geometry of $\mathcal{H}_2$ give an important morphism whose target is $\mathrm{VSP}\,(F_4, n; \mathcal{H}_2)$: see Theorem 2.5.12.

2.5.1. *Special blow-up $\widetilde{A}$ of A.* Similarly to (2.6), we consider the following diagram:

$$\begin{tikzcd}
\mathcal{U}_2' \subset \mathcal{U}_2 \times \mathcal{H}_2 \arrow[r, "{(\psi,\mathrm{id})}"] \arrow[d] & A \times \mathcal{H}_2 \supset \mathcal{U}_2 \arrow[d] \\
\widehat{\mathcal{D}}_2 \subset \mathcal{H}_2 \times \mathcal{H}_2 \arrow[r] & \mathcal{H}_2.
\end{tikzcd} \tag{2.9}$$

Let $\mathcal{U}_2' \subset \mathcal{U}_2 \times \mathcal{H}_2$ be the pull-back of $\mathcal{U}_2$ and $\widehat{\mathcal{D}}_2$ the image of $\mathcal{U}_2'$ on $\mathcal{H}_2 \times \mathcal{H}_2$. Similarly to the investigation of the diagram (2.6), we see that the image $\mathcal{F}'$ in $\mathcal{H}_2 \times \mathcal{H}_2$ of the inverse image of $\cup_{i=1}^n \beta_i' \times \mathcal{H}_2$ is not divisorial nor does not dominate $\mathcal{H}_2$. Moreover, any component of $\widehat{\mathcal{D}}_2$ outside $\mathcal{F}'$ dominates $\mathcal{H}_2$, and is divisorial or possibly the diagonal of $\mathcal{H}_2 \times \mathcal{H}_2$. Note that dislike the diagram (2.6), there is no other non-divisorial component in this case. Compare the proof of Corollary 2.4.14. Here we leave the possibility that the diagonal of $\mathcal{H}_2 \times \mathcal{H}_2$ is contained in the divisorial component of $\widehat{\mathcal{D}}_2$. We, however, prove this is not the case in Lemma 2.5.9.

Let $\mathcal{D}_2 \subset \mathcal{H}_2 \times \mathcal{H}_2$ be the union of the divisorial components of $\widehat{\mathcal{D}}_2$ with reduced structure. $\mathcal{D}_2$ is Cartier since $\mathcal{H}_2 \times \mathcal{H}_2$ is smooth. $\mathcal{D}_2 \to \mathcal{H}_2$ is flat since $\mathcal{D}_2$ is Cohen-Macaulay, $\mathcal{H}_2$ is smooth and $\mathcal{D}_2 \to \mathcal{H}_2$ is equi-dimensional. Let D_q be the fiber of $\mathcal{D}_2 \to \mathcal{H}_2$ over $[q] \in \mathcal{H}_2$ via the projection to the second factor.

We are almost ready to define the modification of $\psi\colon \mathcal{U}_2 \to A$ we are looking for. To find the range we consider the blow-up of A along $\cup_{i=1}^n \beta'_i$ and we denote it by $\rho\colon \widetilde{A} \to A$.

Lemma 2.5.1.

$$\mathcal{N}_{\beta'_i/A} = \mathcal{O}_{\beta'_i}(-1) \oplus \mathcal{O}_{\beta'_i}(-1).$$

Proof. We prove the assertion by using the inductive construction of C_d. The assertion is clear for $d = 1$ since C_1 has no bi-secant line. Suppose the assertion holds for C_{d-1}. Choose a general uni-secant line $\overline{l} \subset B$ of C_{d-1}. Let $\overline{m}_1, \dots, \overline{m}_{d-2}$ be the lines on B intersecting both C_{d-1} and $\overline{l}$ outside $C_{d-1} \cap \overline{l}$. Let $A' \to B$ be the blow-up along $C_{d-1} \cup \overline{l}$. Note that the smoothing $C_{d-1} \cup \overline{l}$ to C_d induces that of A' to A. Let $\widetilde{m}_i$ be the strict transform of $\overline{m}_i$ on A'. By the smoothing construction of C_d from $C_{d-1} \cup \overline{l}$ and the assumption on induction, we have only to prove $\mathcal{N}_{\widetilde{m}_i/A'} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let $A'_1 \to B$ be the blow-up along $\overline{l}$ and $A'_2 \to A'_1$ the blow-up along the strict transform of C_{d-1}. Denote by m'_i and m''_i the strict transform of $\overline{m}_i$ on A'_1 and A'_2 respectively. Then $\mathcal{N}_{\widetilde{m}_i/A'} = \mathcal{N}_{m''_i/A'_2}$. Since m'_i is a fiber of $A'_1 \to Q$ (cf. Proposition 2.1.6 (2)), we have $\mathcal{N}_{m'_i/A'_1} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Let F be the exceptional divisor of $A'_1 \to Q$ and F' the strict transform of F on A'_2. We may suppose F and C'_{d-1} intersect transversely, thus $F' \to F$ is the blow-up at $d-2$ points $m'_i \cap C'_{d-1}$ $(i = 1, \dots, d-2)$. Thus $F' \cdot m''_i = -1$ and $\mathcal{N}_{m''_i/F'} = \mathcal{O}_{\mathbb{P}^1}(-1)$, and this implies the assertion. □

We add the following piece of notation:

Notation 2.5.2. (1) $E_i := \rho^{-1}(\beta'_i)$. By Lemma 2.5.1, $E_i \simeq \mathbb{P}^1 \times \mathbb{P}^1$,
(2) $f_i :=$ a general fiber of $\rho_{|E_i}\colon E_i \to \beta'_i$,
(3) $\gamma_i :=$ a general fiber of the other projection $E_i \to \mathbb{P}^1$,
(4) $\widetilde{E}_C :=$ the strict transform of E_C, and
(5) $\widetilde{\zeta}_{ij} :=$ the strict transform of the fiber ζ_{ij} of E_C over $p_{ij} \in C \cap \beta_i$,
where $i = 1, \dots, s$ and $j = 1, 2$.

The domain of the finite morphism is $\widetilde{\mathcal{U}}_2 := \mathcal{U}_2 \times_A \widetilde{A}$; in other words, $\widetilde{\mathcal{U}}_2$ is the blow-up of $\mathcal{U}_2$ along $\mathcal{U}_2 \cap (\cup_{i=1}^s \beta'_i \times \mathcal{H}_2)$. We obtain that the natural morphism $\widetilde{\mathcal{U}}_2 \to \widetilde{A}$ is finite after an analysis of the morphism $\mathcal{U}_2 \to A$ in the neighborhood of some special conics and via the suitable notion of conic on $\widetilde{A}$.

Note that, by Proposition 2.2.4 (5), there are $d-4$ lines $\alpha_1, \dots, \alpha_{d-4}$ distinct from β_i and intersecting both C and β_i outside $C \cap \beta_i$. Set $t_k := \alpha_k \cap C$. Corresponding to α_k, there are two marked conics $(\alpha_k \cup \beta_i; p_{i1}, t_k)$ and $(\alpha_k \cup \beta_i; p_{i2}, t_k)$, which does not belong to e_i (by the choice of marking). We denote by ξ_{ijk} the conics on A corresponding to $(\alpha_k \cup \beta_i; p_{ij}, t_k)$, where $i = 1, \dots, s$, $j = 1, 2$, and $k = 1, \dots, d-4$.

Lemma 2.5.3. *ξ_{ijk} does not belong to D_{β_i}.*

Proof. By the projection from β_i, the image $\overline{q}$ of a general conic q belonging to D_{β_i} maps to a bi-secant line of the image $C' \subset Q$ of C, and α_k maps to a point p_{ijk}. Let p'_{ij} be the point of C' corresponding to p_{ij}. Let F be the exceptional divisor over β_i, and F' the image of F on Q. We say a ruling of $F' \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is horizontal if it does not come from a fiber of $F \to \beta_i$. If $[\xi_{ijk}] \in D_{\beta_i}$, then ξ_{ijk} corresponds to a bi-secant line of C', which must be the horizontal ruling of F' through p'_{ij} and p_{ijk}. By inductive construction of C, we can prove that p'_{ij} and p_{ijk} do not lie on a horizontal ruling. Thus we have the claim. □

Definition 2.5.4. We say that a curve $\widetilde{q} \subset \widetilde{A}$ is a *conic* on $\widetilde{A}$ if

(i) $\widetilde{q}$ is connected and reduced,
(ii) $-K_{\widetilde{A}} \cdot \widetilde{q} = 2$,
(iii) $\widetilde{E}_C \cdot \widetilde{q} = 2$,
(iv) $E_i \cdot \widetilde{q} = 0$, and
(v) $p_a(\widetilde{q}) = 0$.

Similarly to the case of conics on A, we know there exists a unique two-dimensional component of the Hilbert scheme of $\widetilde{A}$ parameterizing conics on $\widetilde{A}$. Let $\mathcal{H}_2^{\widetilde{A}}$ be the normalization of the two-dimensional component. Similarly to the proof of Proposition 2.4.9, we can show $\mathcal{H}_2^{\widetilde{A}}$ is proper and there is a natural birational morphism $\mathcal{H}_2^{\widetilde{A}} \to \overline{\mathcal{H}}_2$. Since $\mathcal{H}_2 \to \overline{\mathcal{H}}_2$ is the normalization, we have also a natural morphism $\mathcal{H}_2^{\widetilde{A}} \to \mathcal{H}_2$. We do not need a full classification of conics on $\widetilde{A}$ but only the following:

Lemma 2.5.5. (1) *There is a unique conic $\widetilde{q}$ on $\widetilde{A}$ corresponding to a conic q on A belonging to e_i, and moreover, $\widetilde{q}$ is isomorphic to q over the component β'_i. In particular, $\mathcal{H}_2^{\widetilde{A}} \to \mathcal{H}_2$ is isomorphic near e_i.*

(2) *A conic belonging to D_{β_i} is smooth near β'_i. There is a unique conic $\widetilde{q}$ on $\widetilde{A}$ corresponding to a conic q on A belonging to D_{β_i}, and, over β'_i, $\widetilde{q}$ is isomorphic to the union of q and the fiber of E_i over $q \cap \beta'_i$. In particular, $\mathcal{H}_2^{\widetilde{A}} \to \mathcal{H}_2$ is isomorphic near D_{β_i}.*

Proof. This follows from an explicit calculation as in the proof of Proposition 2.3.6. For the first statement of (2), we use Proposition 2.2.2 (5) and Lemma 2.5.3. □

Let $\Gamma := \mathcal{U}_2 \cap (\cup_{i=1}^s \beta'_i \times \mathcal{H}_2)$. Outside $\cup_i \beta'_i \times e_i$, Γ is set-theoretically the disjoint union of

$$\Gamma_i := \{(x, [q]) \mid [q] \in D_{\beta_i}, x \in q \cap \beta'_i\}\ (i = 1, \dots, s),$$

which is a section of μ over D_{β_i}, and

$$\Gamma_{ijk} := \{(x, [\xi_{ijk}]) \mid x \in \beta'_i\}\ (k = 1, \dots, d-4,\ j = 1, 2).$$

Lemma 2.5.6. *Along Γ_{ijk}, $\mathcal{U}_2$ is smooth and Γ is reduced.*

Proof. To show that $\mathcal{U}_2$ is smooth near Γ_{ijk}, we have only to see that the conic ξ_{ijk} is strongly smoothable. Note that $\mathcal{N}_{\beta'_i/A} \simeq \mathcal{O}_{\mathbb{P}_1}(-1)^{\oplus 2}$, $\mathcal{N}_{\alpha'_k/A} \simeq \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$ and $\mathcal{N}_{\zeta_{i,3-j}/A} \simeq \mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1)$. We apply [HH85, Theorem 4.1] by setting $C = \beta'_i$, $D = \alpha'_k \cup \zeta_{i,3-j}$ and $S = (\alpha'_k \cap \beta'_i) \cup (\zeta_{i,3-j} \cap \beta'_i)$. We check the conditions a) and b) of [ibid.]. The condition a) clearly holds. The condition b) follows from the following two facts:

(1) let F be the exceptional divisor of the blow up of B along α_k. Note that $F \simeq \mathbb{P}^1 \times \mathbb{P}^1$. We say a fiber of $F \to \mathbb{P}^1$ in the other direction to $F \to \alpha_k$ a horizontal fiber. Then the intersection points of the strict transform of C and F, and the strict transform of β_i and F do not lie on a common horizontal fiber.

This can be proved by the inductive construction of $C = C_d$ in a similar fashion to the proof of Lemma 2.5.1, and

(2) let G be the exceptional divisor of the blow up of A along $\zeta_{i,3-j}$. Note that $G \simeq \mathbb{F}_1$. Then the intersection points of the strict transform of β'_i and G does not lie on the negative section of G.

This can be easily proved by noting $\zeta_{i,3-j}$ is a fiber of E.

Thus, by [HH85, Theorem 4.1], ξ_{ijk} is strongly smoothable.

Second, we prove that Γ is reduced along Γ_{ijk}. We have only to prove that $\mathcal{U}_2 \to A$ is unramified along Γ_{ijk} since then Γ is the étale pull-back of β'_i near Γ_{ijk}, hence is reduced.

By the inductive construction of $C = C_d$ and the following exact sequence:

$$0 \to \mathcal{N}_{\beta'_i/A} \to \mathcal{N}_{\xi_{ijk}/A|\beta'_i} \to T^1_S \to 0,$$

we can prove that $\mathcal{N}_{\xi_{ijk}/A|\beta'_i} \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$. Thus $H^0(\mathcal{N}_{\xi_{ijk}/A}) \otimes \mathcal{O}_{\xi_{ijk}} \to \mathcal{N}_{\xi_{ijk}/A}$ is surjective at a point of Γ_{ijk} since it factor through the surjection $H^0(\mathcal{N}_{\xi_{ijk}/A|\beta'_i}) \otimes \mathcal{O}_{\beta'_i} \to \mathcal{N}_{\xi_{ijk}/A|\beta'_i}$. Thus $\mathcal{U}_2 \to A$ is unramified along Γ_{ijk}. $\square$

The next proposition contains the finitess result we need.

Proposition 2.5.7. *$\widetilde{\mathcal{U}}_2$ is Cohen-Macaulay and the natural morphism $\widetilde{\psi}\colon \widetilde{\mathcal{U}}_2 \to \widetilde{A}$ is finite (of degree $n := \frac{(d-1)(d-2)}{2}$). In particular, $\widetilde{\psi}$ is flat.*

Proof. Lemma 2.5.5 shows that $\widetilde{\mathcal{U}}_2 \to \mathcal{H}_2$ is isomorphic to the universal family of conics on $\widetilde{A}$ over e_i and D_{β_i}. Thus $\widetilde{\mathcal{U}}_2$ is Cohen-Macaulay over e_i and D_{β_i} since so are the fibers. Note that $\widetilde{\mathcal{U}}_2 \to \mathcal{U}_2$ is the blow-up along Γ_i near Γ_i and is an isomorphism near $\beta'_i \times e_i$.

Lemma 2.5.6 shows that $\widetilde{\mathcal{U}}_2 \to \mathcal{U}_2$ is the blow-up along Γ_{ijk} near $\xi_{ijk} \times [\xi_{ijk}]$, and $\widetilde{\mathcal{U}}_2$ is smooth over Γ_{ijk}. Thus $\widetilde{\mathcal{U}}_2$ is Cohen-Macaulay. To see $\widetilde{\psi}$ is finite, we have only to note that the inverse images of $\beta'_i \times e_i$ on $\widetilde{\mathcal{U}}_2$ and the exceptional divisor of $\widetilde{\mathcal{U}}_2 \to \mathcal{U}_2$ are not contracted by $\widetilde{\psi}$. $\square$

From now on in the section 3, we assume that $d \geq 6$ and we consider $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$.

Consider the following diagram:

(2.10)
$$\begin{tikzcd} & \widetilde{\mathcal{U}_2} \arrow[dl, "\widetilde{\mu}"'] \arrow[dr, "\widetilde{\psi}"] & \\ \mathcal{H}_2 & & \widetilde{A}. \end{tikzcd}$$

Definition 2.5.8. Let $\widetilde{a}$ be a point of $\widetilde{A}$. We say that $[\widetilde{\psi}^{-1}(\widetilde{a})] \in \operatorname{Hilb}^n \check{\mathbb{P}}^{d-3}$ is *the cluster of conics attached to* $\widetilde{a}$ and denote it by $[\mathcal{Z}_{\widetilde{a}}]$. A conic q such that $[q] \in \operatorname{Supp} \mathcal{Z}_{\widetilde{a}}$ is called *a conic attached to* $\widetilde{a}$.

Remark. Though we do not need it later, we describe the fiber of $\widetilde{\psi}$ over a general point $\widetilde{a} \in E_i$ for some i for reader's convenience. In other words, we exhibit n conics attached to $\widetilde{a}$.

Set $a := \rho(\widetilde{a}) \in A$ and $b := f(a) \in \beta_i$. We use notations of Proposition 2.4.13. Since $\deg C_b = d - 2$, the number of bi-secant conics through b not belonging to the family e_i is given by the number of double points of C_b, which is $\frac{(d-3)(d-4)}{2}$. Moreover $2(d-4)$ conics ξ_{ijk} through a.

The number of remaining conics is $3 = n - \frac{(d-3)(d-4)}{2} - 2(d-4)$. Such conics will belong to e_i. By Lemma 2.5.5, $\widetilde{\mathcal{U}}_2 \to \mathcal{H}_2$ is isomorphic to the universal family of conics on $\widetilde{A}$ over e_i. Thus a desired conic on A is the image of a conic $\widetilde{q}$ on $\widetilde{A}$ such that $\widetilde{a} \in \widetilde{q}$ and $\rho(\widetilde{q})$ belongs to e_i. We show there are three such conics. Let S_i be the strict transform on $\widetilde{A}$ of the locus of lines intersecting β_i. Then it is easy to

see that $S_{i|E_i}$ does not contain any fiber γ_i of the second projection $\sigma_i \colon E_i \to \mathbb{P}^1$. Moreover $S_{i|E_i} \sim 2\gamma_i + 3f_i$. Let γ_i' be the fiber of σ_i through $\widetilde{a}$. Then γ_i' intersect S_i at three points. Corresponding to these three points, there are three lines on B intersecting β_i. Denote by l_1, l_2 and $l_3 \subset A$ the strict transforms of these three lines. Then $\beta_i' \cup l_j$ $(j = 1, 2, 3)$ are the conics on A what we want.

By Proposition 2.5.7 and the universal property of Hilbert schemes, we obtain a naturally defined map $\Psi \colon \widetilde{A} \to \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$. This is clearly injective because n conics attached to a point $\widetilde{a} \in \widetilde{A}$ uniquely determines $\widetilde{a}$.

The task is to understand the image of Ψ.

2.5.2. *Morphism from $\widetilde{A}$ to VSP.*

To understand the image of $\Psi \colon \widetilde{A} \to \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3}$ we construct explicitly a quartic polynomial which plays the role of the plane quartic in the Mukai's interpretation of V_{22}.

Lemma 2.5.9. *$\mathcal{D}_2$ does not contain the diagonal of $\mathcal{H}_2 \times \mathcal{H}_2$. In particular we have the following:*

let $\widetilde{a}$ be a general point of $\widetilde{A}$ and $q_1, q_2, \dots, q_n \in \mathcal{H}_2$ the conics attached to $\widetilde{a}$. Then

$$D_{q_i}([q_i]) \neq 0$$

for $1 \leq i \leq n$.

Proof. Here we assume $d \geq 3$. It suffices to prove that $D_q([q]) \neq 0$ for a general $[q] \in \mathcal{H}_2$. This is equivalent to that the image $D_q^\flat$ on $\overline{\mathcal{H}}_2$ of D_q does not contain $[\overline{q}]$. Note that $D_q^\flat$ is the closure of the locus of multi-secant conics of C intersecting properly $\overline{q}$. Now the assertion follows from the inductive construction of C_d from $C_{d-1} \cup \overline{l}$. From now on, we denote $D_q^\flat$ for C_d by $D_{q,d}^\flat$. If $d = 3$, then $D_q \sim 0$, thus the assertion trivially true. If $D_{q',d-1}^\flat([\overline{q}']) \neq 0$ for a general multi-secant conic $\overline{q}'$ of C_{d-1}, then $D_{q,d}^\flat([\overline{q}]) \neq 0$ for a general multi-secant conic $\overline{q}$ of C_d. $\square$

The proof of the following lemma is almost identical to the one of Theorem 2.4.18; then we omit it:

Lemma 2.5.10. $D_q \sim 2(d-3)h - 2\sum_{i=1}^{e} e_i$ *for a conic q, namely, D_q is a quadric section of $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$.*

We proceed to construct the quartic polynomial. By the seesaw theorem, it holds that $\mathcal{D}_2 \sim p_1^* D_q + p_2^* D_q$. Consider the morphism $\mathcal{H}_2 \times \mathcal{H}_2$ into $\check{\mathbb{P}}^{d-2} \times \check{\mathbb{P}}^{d-3}$ defined by $|p_1^* D_l + p_2^* D_l|$, which is an embedding since $d \geq 6$. Since it is easy to see that

$$H^0(\mathcal{H}_2 \times \mathcal{H}_2, \mathcal{D}_2) \simeq H^0(\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}, \mathcal{O}(2,2)),$$

it holds that $\mathcal{D}_2$ is the restriction of a unique $(2,2)$-divisor on $\check{\mathbb{P}}^{d-3} \times \check{\mathbb{P}}^{d-3}$, which we denote by $\{\widetilde{\mathcal{D}}_2 = 0\}$. Since $\{\widetilde{\mathcal{D}}_2 = 0\}$ is also symmetric, we may take the equation $\widetilde{\mathcal{D}}_2$ so that it is the bi-homogenization of an equation $\check{F}_4$ of a quartic in $\check{\mathbb{P}}^{d-3}$ (cf. [DK93, §1]). Moreover the fiber of $\{\widetilde{\mathcal{D}}_2 = 0\}$ over a point $p \in \check{\mathbb{P}}^{d-3}$ is defined by the polar $P_p(\check{F}_4)$, which we denote by $\widetilde{D}_p$. For $[q] \in \mathcal{H}_2$, we denote $\widetilde{D}_{[q]}$ simply by $\widetilde{D}_q$. By construction, $D_q = \{\widetilde{D}_q = 0\} \cap (\mathcal{H}_2 \times \mathcal{H}_2)$. We may choose the defining equation H_q of the hyperplane of $\mathbb{P}^{d-3}$ corresponding to $[q]$ such that $P_{H_q^2}(\check{F}_4) = \widetilde{D}_q$.

From now on, we write $\mathbb{P}^{d-3} = \mathbb{P}_* V$, where V is the $d-2$-dimensional vector space. The crucial point in the following assertions is that the number of the conics attached to a point of $\widetilde{A}$ coincides with $\dim_{\mathbb{C}} S^2 V$.

Let $\widetilde{a}$ be a general point of $\widetilde{A}$ and $q_1, \dots, q_n$ are the conics attached to $\widetilde{a}$. By the definition of $\widetilde{D}_{q_i}$ and generality of $\widetilde{a}$, we have the following (we use Lemma 2.5.9):

$$\widetilde{D}_{q_j}([q_i]) = 0 \ (j \neq i) \text{ and } \widetilde{D}_{q_i}([q_i]) \neq 0. \tag{2.11}$$

(2.11) implies $\widetilde{D}_{q_1}, \dots, \widetilde{D}_{q_n}$ are linearly independent. Thus by $P_{H^2_{q_i}}(\check{F}_4) = \widetilde{D}_{q_i}$, it holds that the apolarity map

$$\begin{array}{rccc} \mathrm{ap}_{\check{F}_4} \colon & S^2\check{V} & \to & S^2 V \\ & G & \mapsto & P_G(\check{F}_4) \end{array}$$

is an isomorphism. Moreover, $H^2_{q_1}, \dots, H^2_{q_n}$ are linearly independent. Thus $\check{F}_4$ is *non-degenerate* in the sense of Dolgachev. By [Dol04, §2.3], there exists a unique quartic form F_4 such that $\mathrm{ap}_{F_4} = \mathrm{ap}^{-1}_{\check{F}_4}$. In particular, it holds

$$P_{\widetilde{D}_q}(F_4) = H^2_q.$$

F_4 is called *the quartic form dual to* $\check{F}_4$.

To see the relation between the set of conics attached to a general point of $\widetilde{A}$ and the representation of F_4 as a sum of powers of linear forms we need to find conditions which force n conics to be attached to $\widetilde{a} \in \widetilde{A}$. Next lemma is sufficient for our purposes.

Lemma 2.5.11. *Let $q_1, \dots, q_n$ be n distinct conics such that*
(1) $\widetilde{D}_{q_i}([q_j]) = 0$ *for all* $i \neq j$,
(2) *all the* $\overline{q}_i$ *are smooth,*
(3) *if three of* $\overline{q}_i$ *pass through a point* b*, then any other* $\overline{q}_i$ *does not intersect a line through* b *outside* b*, and*
(4) *no two of* $\overline{q}_i$ *intersect at a point of* $C \cup \cup_i \beta_i$.

Then the q_i*'s are attached to a point of* $\widetilde{A}$.

Proof. By the assumption (1), $\overline{q}_1, \dots, \overline{q}_n$ are mutually intersecting multi-secant conics of C. By the assumption (4), it suffices to prove $\overline{q}_1, \dots, \overline{q}_n$ pass through one point of B.

Step 1. Let $b \in B$ be a point such that five of $\overline{q}_i$, say, $\overline{q}_1, \dots, \overline{q}_5$ pass through b. Then all the $\overline{q}_i$ pass through b.

By the double projection from b, $\overline{q}_1, \dots, \overline{q}_5$ are mapped to points $p_1, \dots, p_5$ on $\mathbb{P}^2$. Suppose by contradiction that a smooth conic $\overline{q}_j$ does not pass through b. Let q'_j, q''_j and $\widetilde{q}_j$ be the strict transforms of $\overline{q}_j$ on B_b, B'_b and $\mathbb{P}^2$, and set $S := \pi^*_{2b}\widetilde{q}_j$. By the assumption (3), $\overline{q}_j$ does not intersect a line through b. Thus $\widetilde{q}_j$ is a smooth conic through $p_1, \dots, p_5$. The conic $\widetilde{q}_j$ is unique since a conic through five points is unique. It holds that $-K_{B'_b} \cdot q''_j = 4$ and $S \cdot q''_j = 4$, thus $S \simeq \mathbb{F}_2$ and q''_j is the negative section. This implies that q_j is also unique. By reordering, we may assume that $j = n$. We have the configuration such that all the conics pass through b except q_n. Denote by p_i the image of q_i $(i \neq n)$. Then $\widetilde{q}_n$ and C_b intersect at p_i. By $d \geq 6$, it holds $\deg C_b \geq 3$, thus $\widetilde{q}_n \neq C_b$. By the assumption (4), $b \notin C$. Therefore $\widetilde{q}_n$ and C_b intersect at $n-1$ singular points of C_b. Since $\deg C_b \leq d$, it

holds $2(n-1) \leq 2d$, a contradiction.

Step 2. If four conics $\overline{q}_1, \dots, \overline{q}_4$ pass through one point b, then all the conics pass through b.

By contradiction and Step 1, we may assume that all the conics except $\overline{q}_1, \dots, \overline{q}_4$ do not pass through b. Pick up two any conics, say, $\overline{q}_5$ and $\overline{q}_6$, not passing through b. Considering the double projection from b as in Step 1. Denote by $\widetilde{q}_j$ $(j \geq 5)$ the image of $\overline{q}_j$ on $\mathbb{P}^2$. By the assumption (3), $\overline{q}_5$ and $\overline{q}_6$ do not intersect a line through b, thus $\widetilde{q}_5$ and $\widetilde{q}_6$ are conics on $\mathbb{P}^2$. Therefore $\overline{q}_5 \cap \overline{q}_6$ lies on one of $\overline{q}_1, \dots, \overline{q}_4$ since otherwise $\widetilde{q}_5$ and $\widetilde{q}_6$ would intersect at five points and this is a contradiction as in Step 1. Thus any two conics intersect on $\overline{q}_1, \dots, \overline{q}_4$. Let p_i be the intersection $\overline{q}_i \cap \overline{q}_5$ for $i = 1, \dots, 4$. Then $\overline{q}_j$ $(j \geq 5)$ pass through one of p_i. Thus one of p_i, say, p_1, there pass through at least $\lceil \frac{(n-5)}{4} \rceil$ conics. By Step 1, $\lceil \frac{(n-5)}{4} \rceil \leq 2$ (already $\overline{q}_1$ and $\overline{q}_5$ pass through p_1). This implies $d = 6$. We exclude this case in Step 3. Note that if $d = 6$, then the four conics $\overline{q}_1$, $\overline{q}_2$, $\overline{q}_5$, and $\overline{q}_6$ mutually intersect and the all the intersection points are different. By reordering conics, we assume that $\overline{q}_i$ $(1 \leq i \leq 4)$ satisfy this property.

Step 3. We complete the proof.

Assume by contradiction that $\overline{q}_1, \dots, \overline{q}_n$ do not pass through one point on B. If $d \geq 7$, then, by Steps 1 and 2,

(2.12) at most three of $\overline{q}_i$'s pass through any intersection point.

Let m be the number of conics in a maximal tree T of $\overline{q}_i$'s such that two conics in T pass through any intersection point. Note that T is connected since $\overline{q}_i$'s mutually intersect. The number of the intersection points of $\overline{q}_i$'s contained in T is $\frac{m(m-1)}{2}$.

By the maximality of T, a conic not belonging to T passes through one of the intersection points of conics in T. By (2.12), no two conics not belonging to T pass through one of the intersection point of conics in T. Hence it holds $\frac{m(m-1)}{2} + m \geq n$. This implies that $m \geq d-2$ by $n = \frac{(d-1)(d-2)}{2}$. By reordering, we assume that $\overline{q}_1, \dots, \overline{q}_m$ belong to T. If $d = 6$, then we take $\overline{q}_1, \dots, \overline{q}_4$ as in the last part of Step 2. Consider the projection $B \dashrightarrow \mathbb{P}^3$ from $\overline{q}_1$. Then $\overline{q}_2, \dots, \overline{q}_m$ are mapped to lines $l_2, \dots, l_m$ intersecting mutually on $\mathbb{P}^3$ and the intersection points are different. Thus $l_2, \dots, l_m$ span a plane, which in turn shows that $\overline{q}_1, \dots, \overline{q}_m$ span a hyperplane section H on B. Since C intersects $\overline{q}_i$ at two point or more, C intersects H at $2m$ points or more by the assumption (4). But $2m \geq 2(d-2) > d$, C must be contained in H, a contradiction to Proposition 2.2.1 (d). $\square$

We think the next theorem to be of theoretical relevance in itself and as a first result to understand varieties of sum of powers confined in a subvariety.

Theorem 2.5.12. $\operatorname{Im}\Phi$ *is an irreducible component of* $\mathrm{VSP}\,(F_4, n; \mathcal{H}_2)$.

Proof. Set

$$Z := \{([H_1], \dots, [H_n]) \in \mathrm{Hilb}^n \check{\mathbb{P}}^{d-3} \mid H_1^4 + \dots + H_n^4 = F_4, [H_i] \in \mathcal{H}_2\}.$$

For a general point $\widetilde{a}$ and conics $q_1, \dots, q_n$ attached to $\widetilde{a}$, we have (2.11). Conversely, n conics q_i satisfying (2.11) and the assumptions (2)–(4) of Lemma 2.5.11 determine a point of $\widetilde{A}$. Note that the assumptions (2)–(4) of Lemma 2.5.11 are open conditions. Thus we have only to prove that (2.11) is equivalent to

$$\alpha_1 H_{q_1}^4 + \dots + \alpha_n H_{q_n}^4 = F_4 \text{ with some nonzero } \alpha_i \in \mathbb{C}. \tag{2.13}$$

We see that (2.13) is equivalent to

(2.14) If $\{G=0\}\subset\check{\mathbb{P}}^{d-3}$ is any quartic through $[q_1],\cdots,[q_n]$, then $P_{F_4}(G)=0$.

Indeed, by the apolarity pairing, $\langle G, H^4_{q_i}\rangle=0\Leftrightarrow G([q_i])=0$, thus, the assumption on G is equivalent to $G\subset\langle H^4_{q_1},\dots,H^4_{q_n}\rangle^{\perp}$. Therefore (2.13) is equivalent to $\langle H^4_{q_1},\dots,H^4_{q_n}\rangle^{\perp}\subset\langle F_4\rangle^{\perp}$. Since F_4 is non-degenerate, this is equivalent to (2.13).

We show (2.11) implies (2.14). If (2.11) holds, then $\widetilde{D}_{q_i}$ $(i\neq 1)$ generate the space of linear forms passing through $[q_1]$, we may write $G=Q_2\widetilde{D}_{q_2}+\cdots+Q_n\widetilde{D}_{q_n}$, where Q_i are quadratic forms on $\check{\mathbb{P}}^{d-3}$. By $G([q_i])=0$ for $i\neq 1$, we have $Q_i([q_i])\widetilde{D}_{q_i}([q_i])=0$. $\widetilde{D}_{q_i}([q_i])\neq 0$ implies that $Q_i([q_i])=0$. Thus Q_i is a linear combination of $\widetilde{D}_{q_j}$ $(j\neq i)$. Consequently, G is a linear combination of $\widetilde{D}_{q_i}\widetilde{D}_{q_j}$ $(1\le i<j\le n)$. Thus $P_{F_4}(G)=0$ follows from that

$$P_{F_4}(\widetilde{D}_{q_i}\widetilde{D}_{q_j})=P_{H_{q_i}}(\widetilde{D}_{q_j})=\widetilde{D}_{q_j}([q_i])=0.$$

Finally we show (2.13) implies (2.11). By (2.13), it holds

$$H^2_{q_i}=P_{\widetilde{D}_{q_i}}(F_4)=\sum\alpha_j\langle\widetilde{D}_{q_i},H^4_{q_j}\rangle H^2_{q_j}.$$

Since $H^2_{q_j}$ are linearly independent, (2.11) holds. □

Definition 2.5.13. We say $\operatorname{Im}\Phi$ is the *main component* of $\mathrm{VSP}\,(n,F_4;\mathcal{H}_2)$.

The following lemma characterizes the main component of $\mathrm{VSP}\,(n,F_4;\mathcal{H}_2)$, which will play a crucial role in 3.7:

Lemma 2.5.14. *Let $(\mathcal{H}_2^k)^o$ and $(\mathrm{Hilb}^k\check{\mathbb{P}}^{d-3})^o$ $(k\in\mathbb{N})$ be the complements of all the small diagonals of $\mathcal{H}_2^k$ (k times product of $\mathcal{H}_2$) and $\mathrm{Hilb}^k\check{\mathbb{P}}^{d-3}$ respectively. Set*

$$\mathrm{VSP}^{\,o}(F_4,n;\mathcal{H}_2):=\{([H_1],\dots,[H_n])\mid [H_i]\in\mathcal{H}_2, H_1^m+\cdots+H_n^m=F_4\}.$$

Let V^o be the inverse image of $\mathrm{VSP}^{\,o}(F_4,n;\mathcal{H}_2)$ by the natural projection $(\mathcal{H}_2^n)^o\to(\mathrm{Hilb}^n\check{\mathbb{P}}^{d-3})^o$. Let $(\mathcal{H}_2^n)^o\to(\mathcal{H}_2^2)^o$ be the projection to any of two factors. Then a component of V^o dominating $\mathcal{D}_2$ dominates the main component of $\mathrm{VSP}\,(F_4,n;\mathcal{H}_2)$.

Proof. Let $([q_1],[q_2])\in\mathcal{D}_2\cap(\mathcal{H}_2^2)^o$ be a general point and $\{q_i\}$ $(i=1,\dots,n)$ any set of mutually conjugate n conics including q_1 and q_2. Since q_1 and q_2 are general, we may assume that all the q_i are general. By Lemma 2.5.11 and Theorem 2.5.12, it suffices to prove that $q_1,\dots,q_n$ satisfies the conditions (2)–(4) of Lemma 2.5.11.

(2). Let $\overline{r}_1$ and $\overline{r}_2$ are mutually intersecting smooth conics on B and $\overline{r}_3$ a line pair on B intersecting both $\overline{r}_1$ and $\overline{r}_2$. Since the Hilbert scheme of conics on B is 4-dimensional, the pair of $\overline{r}_1$ and $\overline{r}_2$ depends on 7 parameters. If we fix $\overline{r}_1$ and $\overline{r}_2$, then $\overline{r}_3$ depends on 1 parameter. Thus the configuration $\overline{r}_1$, $\overline{r}_2$, $\overline{r}_3$ depends on 8 parameters. Fix $\overline{r}_1$, $\overline{r}_2$ and $\overline{r}_3$. We count the number of parameters of C_d such that C_d intersects each of $\overline{r}_i$ $(i=1,2,3)$ twice. The number of parameters is $h^0((\mathcal{O}_{\mathbb{P}^1}(d-1)\oplus\mathcal{O}_{\mathbb{P}^1}(d-1))\otimes\mathcal{O}_{\mathbb{P}^1}(-6))+6=2d-12+6=2d-6$, where $+6$ means the sum of the numbers of parameters of two points on $\overline{r}_i$ $(i=1,2,3)$. By $2d-6+8=2d+2$, a general C_d has 2-dimensional pairs of mutually intersecting bi-secant conics which intersect at least one bi-secant line pair of C_d. Thus general pairs of mutually intersecting bi-secant conics of C_d, which form a 3-dimensional family, do not intersect a bi-secant line pair of C_d.

(3). Assume by contradiction that $\overline{q}_i$, $\overline{q}_j$ and $\overline{q}_k$ pass through a point b, and $\overline{q}_l$ does not pass through b but intersects a line through b. Then by the double

projection from b, $\overline{q}_l$ is mapped to a line through the three singular points of the image of C_b corresponding to $\overline{q}_i$, $\overline{q}_j$ and $\overline{q}_k$. Thus we have only to prove that for a general point of b on B, three double points of the image of C_b do not lie on a line.

Fix a general point $b \in B$. Let $\overline{r}_1, \overline{r}_2, \overline{r}_3$ be three conics on B through b such that by the double projection from b, they are mapped to three colinear points on $\mathbb{P}^2$. The number of parameters of C_d's intersecting each of $\overline{r}_i$ twice is $h^0((\mathcal{O}_{\mathbb{P}^1}(d-1)\oplus\mathcal{O}_{\mathbb{P}^1}(d-1))\otimes\mathcal{O}_{\mathbb{P}^1}(-6)) = 2d-12$ since $h^1((\mathcal{O}_{\mathbb{P}^1}(d-1)\oplus\mathcal{O}_{\mathbb{P}^1}(d-1))\otimes\mathcal{O}_{\mathbb{P}^1}(-6)) = 0$ Note that the number of parameters of $\overline{r}_1$, $\overline{r}_2$, $\overline{r}_3$ is 5 since that of lines in $\mathbb{P}^2$ is 2, and that of three points on a line is 3. Thus the number of parameters of C_d's such that its image of the double projection from b has three colinear double points is at most $2d-1$. Hence a general C_d does not satisfy this property.

(4). Let r_1 and r_2 be a general pair of mutually conjugate conics on A such that $\overline{r}_1$ and $\overline{r}_2$ are smooth, and $\overline{r}_1$ and $\overline{r}_2$ intersect at a point on $C\cup\cup_i\beta_i$. Such general pairs of conics r_1 and r_2 form a two-dimensional family since $\dim C\cup\cup_i\beta_i = 1$ and if one point t of $C\cup\cup_i\beta_i$ is fixed, then such pairs of conics such that $t \in \overline{r}_1\cap\overline{r}_2$ form a one-dimensional family. For a general pair of r_1 and r_2, the number of the sets of n mutually conjugate conics including r_1 and r_2 is finite since D_{r_1} and D_{r_2} has no common component. Thus $\{q_i\}$ does not contain such a pair by generality whence $\{q_i\}$ satisfies (4). $\square$

2.5.3. *Relation with Mukai's result.* Here we sketch how the argument goes on if $d = 5$ and explain a relation of our result with Theorem 1.2.1.

Assume that $d = 5$. Associated to the birational morphism $\mathcal{H}_2 \to \check{\mathbb{P}}^2$, there exists a non finite birational morphism

$$\Phi\colon \widetilde{A} \to V_{22} := \mathrm{VSP}\,(F_4, 6) \subset \mathrm{Hilb}^6\check{\mathbb{P}}^2,$$

which fits into the following diagram:

$$\begin{tikzcd}
 & & \widetilde{A} \arrow[dl, "\rho"'] \arrow[dr, "\rho'"] \arrow[ddrr, bend left, "\Phi"] & & \\
 & A \arrow[rr, dashed] \arrow[dl, "f"'] & & A' \arrow[dr, "f'"] & \\
B & & & & V_{22},
\end{tikzcd}$$

where

- V_{22} is a smooth prime Fano threefold of genus twelve,
- ρ' is the blow-down of the three ρ-exceptional divisors E_i $(i = 1, 2, 3)$ over the strict transform β'_i in the other direction. In other words, $A \dashrightarrow A'$ is the flops of β'_1, β'_2 and β'_3 (cf. Lemma 2.5.1), and
- the morphism f' contracts the strict transform of the unique hyperplane section S containing C (see Proposition 2.2.1 (d)) to a general line on V_{22}.

The rational map $V_{22} \dashrightarrow B$ is the famous double projection of V_{22} from a general line m first discovered by Iskovskih (see [Isk78]).

We explain how f' and ρ' are interpreted in our context. As we remarked after the proof of Theorem 2.4.18, the morphism $\mathcal{H}_2 \to \check{\mathbb{P}}^2$ defined by $|D_l|$ contracts three curves D_{e_i} which parameterize conics intersecting β'_i. By noting S is covered

by the images of such conics, this corresponds to that the morphism f' contracts the strict transform of S.

We can see that any conic on A except one belonging to D_{e_i} corresponds to that on V_{22} in the usual sense, and the component of Hilbert scheme of V_{22} parameterizing conics is naturally isomorphic to $\check{\mathbb{P}}^2$. The three conics on V_{22} corresponding to the images of D_{e_i} are $\beta''_i \cup m$, where β''_i are the images of the flopped curve corresponding to β'_i.

Let $a \in E_i$. Then six conics on A attached to a are ξ_{ij1} $(j = 1, 2)$, a conic q_a from D_{e_i} and three conics from e_i (see the remark at the end of 2.5.1). Moreover, if a moves in a fiber γ of the other projection $E_i \to \mathbb{P}^1$, then only the conic q_a from D_{e_i} varies. By the contraction $\mathcal{H}_2 \to \check{\mathbb{P}}^2$, there is no difference among points on γ. This is the meaning of the contraction ρ' of E_i in the other direction.

Finally we remark that $\mathcal{H}_1$ is also naturally isomorphic to the component of Hilbert scheme of V_{22} parameterizing lines.

3. The existence of the Scorza quartic

In this section we will use the geometries of $\mathcal{H}_1$ and $\mathcal{H}_2$ to give an affirmative answer to the conjecture of Dolgachev and Kanev [DK93, Introduction p. 218] (see Theorem 3.5.3).

3.1. Theta-correspondence on $\mathcal{H}_1 \times \mathcal{H}_1$.

In this subsection, we regard $\mathcal{H}_1$ as the component of the Hilbert scheme of A parameterizing lines on A.

We will define a non-effective theta characteristic on $\mathcal{H}_1$ by investigating the following set:

$$I := \{([l_1], [l_2]) \in \mathcal{H}_1 \times \mathcal{H}_1 \mid l_1 \text{ and } l_2 \text{ intersect}\}.$$

We need a more precise and technical definition of I. First we reconsider the desingularization morphism $\pi_{|\mathcal{H}_1} \colon \mathcal{H}_1 \to M \subset \mathbb{P}^2$; see Corollary 2.3.2.

Lemma 3.1.1. $h^0(\mathcal{H}_1, (\pi_{|\mathcal{H}_1})^*\mathcal{O}_M(1)) = 3$.

Proof. Let $h \colon S \to \mathcal{H}_1^B \simeq \mathbb{P}^2$ be the blow-up of $\mathcal{H}_1^B$ at the $s = \frac{(d-2)(d-3)}{2}$ nodes of M. Then $\mathcal{H}_1 \sim d\lambda - 2\sum_{i=1}^s \varepsilon_i$, where λ is the pull-back of a general line and ε_i are exceptional curves. By the exact sequence

$$0 \to \mathcal{O}_S(\lambda - \mathcal{H}_1) \to \mathcal{O}_S(\lambda) \to \mathcal{O}_{\mathcal{H}_1}((\pi_{|\mathcal{H}_1})^*\mathcal{O}_M(1)) \to 0$$

together with $h^0(\mathcal{O}_S(\lambda)) = 3$ and $h^0(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = h^1(\mathcal{O}_S(\lambda)) = 0$, we see that $h^0(\mathcal{H}_1, (\pi_{|\mathcal{H}_1})^*\mathcal{O}_M(1)) = 3$ is equivalent to $h^1(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$. By the Riemann-Roch theorem, we have $\chi(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$. Thus by $h^0(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$, $h^1(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$ is equivalent to $h^2(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = 0$. By the Serre duality, $h^2(\mathcal{O}_S(\lambda - \mathcal{H}_1)) = h^0(\mathcal{O}_S((d-4)\lambda - \sum_{i=1}^s \varepsilon_i)$. Thus we have only to prove that there exists no plane curve of degree $d-4$ through s nodes of M. We prove this fact by using the inductive construction of $C = C_d$. In case $d = 2$, the assertion is obvious. From now on in the proof, we put the suffix d to the object depending on d. For example, $s_d := \frac{(d-2)(d-3)}{2}$. Assuming $h^0(\mathcal{O}_{S_d}((d-4)\lambda_d - \sum_{i=1}^{s_d} \varepsilon_{i,d}) = 0$, we prove $h^0(\mathcal{O}_{S_{d+1}}((d-3)\lambda_{d+1} - \sum_{i=1}^{s_{d+1}} \varepsilon_{i,d+1}) = 0$.

Recall that we constructed C_{d+1} by the smoothing of the union of C_d and a general uni-secant line $\bar{l}$ of C_d. By a standard degeneration argument, we have only to prove that there exists no plane curve of degree $d-3$ through s_{d+1} nodes of

$M_d \cup M(\bar{l})$, where s_d of s_{d+1} nodes are those of C_d and the remaining $s_{d+1}-s_d = d-2$ nodes are $M_d \cap M(\bar{l})$ except the two points corresponding to the two other lines $\bar{l}'$, $\bar{l}''$ through $C_d \cap \bar{l}$. Assume that there exists a plane curve G of degree $d-3$ through s_{d+1} nodes of $M_d \cup M(\bar{l})$. Then $G \cap M(\bar{l})$ contains at least $d-2$ points. Since $\deg G = d-3$, this implies $M(\bar{l}) \subset G$. Thus there exists a plane curve of degree $d-4$ through s_d nodes of M_d, a contradiction. □

We denote by δ the g^1_3 on $\mathcal{H}_1$ which defines $\varphi_{|\mathcal{H}_1} \colon \mathcal{H}_1 \to C$. Let l, l' and l'' be three lines on A such that $[l]+[l']+[l''] \sim \delta$. Then $\bar{l}$, $\bar{l}'$ and $\bar{l}''$ are lines through one point of C. Set

$$\theta := (\pi_{|\mathcal{H}_1})^* \mathcal{O}_M(1) - \delta.$$

Let l be any line on A and l', l'' lines such that $[l]+[l']+[l''] \sim \delta$. By $\theta + [l] = \pi^*_{|\mathcal{H}_1}\mathcal{O}_M(1) - [l'] - [l'']$ and Lemma 3.1.1, we have $h^0(\mathcal{H}_1, \mathcal{O}_{\mathcal{H}_1}(\theta + [l])) = 1$. Let $p_i \colon \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_1$ $(i = 1, 2)$ be the two projections and Δ the diagonal of $\mathcal{H}_1 \times \mathcal{H}_1$. Set $\mathcal{L} := \mathcal{O}_{\mathcal{H}_1 \times \mathcal{H}_1}({p_2}^*\theta + \Delta)$. By $h^0(\mathcal{H}_1, \mathcal{O}_{\mathcal{H}_1}(\theta + [l])) = 1$ for any $[l] \in \mathcal{H}_1$, we see that $p_{1*}\mathcal{L}$ is an invertible sheaf. Define an ideal sheaf $\mathcal{I}$ by ${p_1}^* p_{1*}\mathcal{L} = \mathcal{L} \otimes \mathcal{I}$. $\mathcal{I}$ is an invertible sheaf and let I be the divisor defined by $\mathcal{I}$. Then we can extract the following definition:

Definition 3.1.2. I is called the *theta-correspondence*. We will denote by $I([l])$ the fiber of $I \to \mathcal{H}_1$ over $[l]$.

The following result is a generalization of Mukai's result [Muk04, §4, Theorem] in our setting:

Proposition 3.1.3. *θ is a non-effective theta characteristic.*

Proof. By invoking [DK93, Lemma 7.2.1] and the definition of I, it suffices to prove the following:

(a) $h^0(\mathcal{H}_1, \mathcal{O}_{\mathcal{H}_1}(\theta + [l])) = 1$ for any $[l] \in \mathcal{H}_1$,
(b) I is reduced,
(c) I is disjoint from the diagonal,
(d) I is symmetric, and
(e) I is a $(g(\mathcal{H}_1), g(\mathcal{H}_1))$-correspondence.

Let l be any line on A and l', l'' lines such that $[l]+[l']+[l''] \sim \delta$.

We have proved (a) already.

Noting that the line in $\mathbb{P}^2$ joining $[\bar{l}']$ and $[\bar{l}'']$ parameterizes the lines on B intersecting $\bar{l}$, we see that the fiber of $I \to \mathcal{H}_1$ over a general $[l]$ is reduced. Hence I is reduced.

We prove (c). It is equivalent to show that the support of $I([l])$ does not contain $[l]$. By definition $\theta + [l] = \pi^*_{|\mathcal{H}_1}\mathcal{O}_M(1) - [l'] - [l'']$. If $\bar{l}$ is a uni-secant and is not special, then $M(\bar{l})$ does not contain $[\bar{l}]$, thus we are done. If $\bar{l}$ is special, then, by Propositions 2.1.3 (4) and 2.2.4 (2), we are done. If $\bar{l}$ is a bi-secant then by Proposition 2.2.2 (4), we are done.

We prove (d). Let m be a line on A such that $[m]$ is contained in the support of $I([l])$. It suffices to prove that for a general l, $[l]$ is contained in the support of $I([m])$. For a general l, we may assume that $m \neq l'$ or l''. Then it is easy to verify this fact.

Finally we prove (e). Since I is symmetric and $\deg(\theta + [l]) = d-2 = g(\mathcal{H}_1)$, the divisor is a $(g(\mathcal{H}_1), g(\mathcal{H}_1))$-correspondence. □

3.2. **Duality between $\mathcal{H}_1$ and $\mathcal{H}_2$.**

Denote by $\mathbb{P}^{d-3}$ the projective space dual to $\check{\mathbb{P}}^{d-3}$. The family

$$\begin{tikzcd} \mathcal{D}_1 \arrow[r] \arrow[d] & \mathcal{H}_2 \times \mathcal{H}_1 \arrow[dl] \\ \mathcal{H}_1 & \end{tikzcd}$$

induces the morphism

$$\begin{array}{ccc} \mathcal{H}_1 & \to & \mathbb{P}^{d-3} \\ [l] & \mapsto & [D_l]. \end{array}$$

by the universal property of the Hilbert scheme. Since $D_l \neq D_{l'}$ for $l \neq l'$, $\mathcal{H}_1 \to \mathbb{P}^{d-3}$ is injective.

Consider the projection $\mathcal{D}_1 \to \mathcal{H}_2$ and denote by $\widetilde{H}_q$ the fiber over $[q]$. Since $\mathcal{D}_1$ is a Cartier divisor in a smooth 3-fold $\mathcal{H}_1 \times \mathcal{H}_2$ then $\mathcal{D}_1$ is Cohen-Macaulay. Since no conic on A intersects infinitely many lines on A, $\mathcal{D}_1 \to \mathcal{H}_2$ is finite. Then $\mathcal{D}_1 \to \mathcal{H}_2$ is flat since $\mathcal{H}_2$ is smooth. Note that for a general q, $\widetilde{H}_q$ parameterizes all the lines intersecting q. By considering the morphism $\pi_{|\mathcal{H}_1} \colon \mathcal{H}_1 \to M \subset \mathbb{P}^2$, it is easy to see that for a general conic q, $\widetilde{H}_q \in |\pi^*\mathcal{O}_M(2) - 2\delta|$, namely, $\widetilde{H}_q \sim 2\theta \sim K_{\mathcal{H}_1}$. By the flatness of $\mathcal{D}_1 \to \mathcal{H}_2$, it holds $\widetilde{H}_q$ for any q.

Recall that we denote by $\{H_q = 0\}$ the hyperplane in $\mathbb{P}^{d-3}$ corresponding to $[q] \in \check{\mathbb{P}}^{d-3}$. Note that, for $[l] \in \mathcal{H}_1$ and $[q] \in \mathcal{H}_2$, $[l] \in \{H_q = 0\}$ if and only if $D_l([q]) = 0$. Thus $\widetilde{H}_q = \{H_q = 0\}$. Consequently, the injection $\mathcal{H}_1 \to \mathbb{P}^{d-3}$ is the canonical embedding $\Phi_{|K_{\mathcal{H}_1}|} \colon \mathcal{H}_1 \to \mathbb{P}^{d-3}$ by $\widetilde{H}_q \sim K_{\mathcal{H}_1}$.

In case $d = 5$, a similar construction gives the duality of the canonical embedding $\mathcal{H}_1 \subset \mathbb{P}^2$ and the birational morphism $\mathcal{H}_2 \to \check{\mathbb{P}}^2$.

3.3. **Discriminant locus.**

We follow [DK93, 7.1.4 p.279]. Let $\Gamma \subset \mathbb{P}^{g-1}$ be a canonical curve of genus g and θ' a non-effective even theta characteristic on Γ. By the Riemann-Roch theorem, it holds that $h^0(\theta' + x) = 1$ for a point $x \in \Gamma$. Let

$$I := \{(x, y) \mid y \text{ is in the support of the unique member of } |\theta' + x|\} \subset \Gamma \times \Gamma.$$

We call this the *theta-correspondence*, which is consistent with Definition 3.1.2. We denote by $I(x)$ the fiber of $I \to \Gamma$ over x and call it the *theta-polyhedron* attached to x. In other words, $I(x)$ is the unique member of $|\theta' + x|$ as a divisor.

Since the linear hull $\langle I(x) - y\rangle$ is a hyperplane of $\mathbb{P}^{g-1}$, then we can define a morphism $\pi_{\theta'} \colon I \to |K_\Gamma| = \check{\mathbb{P}}^{g-1}$ as a composition of the embedding $I \hookrightarrow \Theta_\Gamma$ and the Gauss map $\gamma \colon \Theta_\Gamma^{\mathrm{ns}} \to \check{\mathbb{P}}^{g-1}$,

where $\Theta_\Gamma \subset J(\Gamma)$ is the theta divisor and $\Theta_\Gamma^{\mathrm{ns}}$ is the nonsingular locus of Θ_Γ.

Definition 3.3.1. The image $\Gamma(\theta')$ of the above morphism $\pi_{\theta'} \colon I \to \check{\mathbb{P}}^{g-1}$ is called the *discriminant locus* of (Γ, θ').

Set-theoretically $\pi_{\theta'}$ is the map $(x, y) \mapsto \langle I(x) - y\rangle$. The hyperplane $\langle I(x) - y\rangle$ is called the *face* of $I(x)$ opposed to y.

From now on in the section 4, we assume that $d \geq 6$ for the pair $(\mathcal{H}_1, \theta)$ and we consider $\mathcal{H}_2 \subset \check{\mathbb{P}}^{d-3}$.

For the pair $(\mathcal{H}_1, \theta)$, we can interpret $\Gamma(\theta)$ by the geometry of lines and conics on A as follows:

Proposition 3.3.2. *For the pair $(\mathcal{H}_1, \theta)$, the discriminant locus $\Gamma(\theta)$ is contained in $\mathcal{H}_2$, and the generic point of the curve $\Gamma(\theta)$ parameterizes line pairs.*

Proof. Take a general point $([l_1], [l_2]) \in I$, equivalently, take two general intersecting lines l_1 and l_2. $l_1 \cup l_2$ is a line pair and the lines corresponding to the points of $I([l_1]) - [l_2]$ are lines intersecting l_1 except l_2. Thus by 3.2, the point in $\check{\mathbb{P}}^{d-3}$ corresponding to the hyperplane $\langle I([l_1]) - [l_2]\rangle$ is nothing but $[l_1 \cup l_2] \in \mathcal{H}_2$. This implies the assertion. $\square$

Proposition 3.3.3. *The curve $\Gamma(\theta)$ belongs to the linear system $|3(d-2)h - 4\sum_{i=1}^{s} e_i|$ on $\mathcal{H}_2$.*

Proof. We can write:

$$\Gamma(\theta) \sim ah - \sum m_i e_i,$$

where $a \in \mathbb{Z}$ and $m_i \in \mathbb{Z}$. For a general $b \in C$, L_b intersects $\Gamma(\theta)$ simply. Thus a is the number of line pairs whose images on B pass through b. There exists three lines l_1, l_2 and l_3 through b. It suffices to count the number of reducible conics on B having one of l_i as a component except $l_1 \cup l_2$, $l_2 \cup l_3$ and $l_3 \cup l_1$. Thus $a = 3(d-2)$.

We will count the number of line pairs belonging to e_i. Each of such line pairs is of the form $l_{ij;k} \cup l_{ij}$, where $l_{ij;k}$ $(k = 1, 2)$ is the strict transform of the line through p_{ij} distinct from β_i. Thus the number of such pairs is four and $m_i \geq 4$.

Finally we will count the number of line pairs intersecting a general line l. By Corollary 2.4.19, D_l does not contain any line pair $l \cup l'$. Since the number of lines on A intersecting a fix line on A is $d-2$, we see that $D_l \cdot \Gamma(\theta) \geq (d-2)(d-3)$. Then

$$(d-2)(d-3) \leq \Gamma(\theta) \cdot D_l = (d-3)a - \sum_{i=1}^{s} m_i.$$

where $s = \frac{(d-2)(d-3)}{2}$. This implies that $m_i = 4$.

$\square$

Corollary 3.3.4. *For $(\mathcal{H}_1, \theta)$, it holds that $\deg \Gamma(\theta) = g(g-1)$ and $p_a(\Gamma(\theta)) = \frac{3}{2}g(g-1) + 1$.*

Proof. The invariants of $\Gamma(\theta)$ are easily calculated. $\square$

3.4. **Definition of the Scorza quartic.**

By Definition 3.3.1, we have the following diagram:

(3.1)
$$\begin{tikzcd}
 & I \subset \Gamma \times \Gamma \arrow[dl, "\pi_{\theta'}"'] \arrow[dr, "p"] & \\
\Gamma(\theta') \subset \check{\mathbb{P}}^{g-1} & & \Gamma \subset \mathbb{P}^{g-1}.
\end{tikzcd}$$

We can define:

$$\overline{D}_H := \pi_{\theta' *} p^*(H \cap \Gamma),$$

where H is an hyperplane of $\mathbb{P}^{g-1}$. It is easy to see:

$$\deg \overline{D}_H = 2g(g-1).$$

Let $S^m\check{V}$ the space of m-th symmetric forms on the vector space V. Note that an element of $S^m\check{V}$ defines a hypersurface of degree m in $\mathbb{P}_*V$. Let $F \in S^{2k}\check{V}$ be a non-degenerate homogeneous form of degree $2k$ and $\check{F} \in S^{2k}V$ the dual homogeneous

form to F defined as in [Dol04, §2.3]. Following [Dol04, 4.1], we define the variety of *the conjugate pairs*

$$\mathrm{CP}(F) := \{([H_1],[H_2]) \in \mathbb{P}_*\check{V} \times \mathbb{P}_*\check{V} \mid \langle H_1^k, P_{H_2^k}(\check{F})\rangle = 0\},$$

where $\langle\ ,\ \rangle$ is the polarity pairing. Let

$$\Delta := \mathrm{CP}(F) \cap (\text{the diagonal of } \mathbb{P}_*\check{V} \times \mathbb{P}_*\check{V}).$$

Since the diagonal of $\mathbb{P}_*\check{V} \times \mathbb{P}_*\check{V}$ is isomorphic to $\mathbb{P}_*\check{V}$ then $\Delta \simeq \{\check{F} = 0\}$.

Set $D'_H := P_{H^k}(\check{F})$ for a hyperplane $H \subset \mathbb{P}_*V$. Then we can write:

$$\mathrm{CP}(F) = \{([H_1],[H_2]) \in \mathbb{P}_*\check{V} \times \mathbb{P}_*\check{V} \mid D'_{H_2}([H_1]) = 0\}.$$

Definition 3.4.1. A non-degenerate quartic $\{F'_4 = 0\}$ is called the *Scorza quartic* for (Γ, θ') if $\{D'_H = 0\} \cap \Gamma(\theta) = \overline{D}_H$ for a hyperplane $\{H = 0\}$ such that $\Gamma \cap \{H = 0\}$ is reduced, where D'_H is defined as above for F'_4.

3.5. **Dolgachev-Kanev's conjecture on the existence of the Scorza quartic.** We show that the following properties hold for general pairs of canonical curves Γ and even theta characteristics θ' as Dolgachev and Kanev conjectured.

(A1) The number of theta-polyhedrons having a general face in common is two. Equivalently, the degree of the map $I \to \Gamma(\theta')$ is two,
(A2) $\Gamma(\theta')$ is not contained in a quadric, and
(A3) I is reduced.

By [DK93, Theorem 9.3.1], these three conditions are sufficient for the existence of the Scorza quartic for the pair (Γ, θ').

First we show that for our trigonal curve $\mathcal{H}_1$ and the even theta characteristic θ defined by intersecting lines the above conditions hold.

Lemma 3.5.1. $(\mathcal{H}_1, \theta)$ *satisfies* (A1)–(A3).

Proof. (A1) This condition means that for general lines l and l' on A such that $([l],[l']) \in I$ the face $\langle I([l]) - [l']\rangle$ belongs only to $I([l])$ and to $I([l'])$.

By contradiction assume that there exists a line m on A such that $m \neq l$, $m \neq l'$ and $\langle I([l]) - [l']\rangle$ is a face of $I([m])$. Then some $d-3$ points of $I([m])$ lie on the hyperplane $\langle I([l]) - [l']\rangle$, equivalently, m intersects $d-3$ lines on A corresponding to the points of $I([l]) \cup I([l'])$ except l and l'. By $d \geq 6$, it holds that, for l or l', say, l, there exist two lines intersecting both l and m. Consider the projection $B \dashrightarrow Q$ from $f(l) = \overline{l}$:

$$\begin{tikzcd} & B_{\overline{l}} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ B & & Q. \end{tikzcd}$$

We use the notation of Proposition 2.1.6 (2). Now notice that by generality of l, $\overline{l} \neq \overline{m} := f(m)$ is equivalent to have $l \neq m$. Since there exist two lines intersecting both $\overline{l}$ and $\overline{m}$, we have $\overline{l} \cap \overline{m} = \emptyset$. Thus the strict transform $\overline{m}'$ of $\overline{m}$ on Q is a line. Since there exist two lines intersecting both $\overline{l}$ and $\overline{m}$, $\overline{m}'$ intersects the image $E'_{\overline{l}}$ of $E_{\overline{l}}$ at two points. Since $E'_{\overline{l}}$ is a hyperplane section on Q, this implies that $\overline{m}' \subset E'_{\overline{l}}$, a contradiction.

(A2) This condition is satisfied by Theorem 2.4.18 (4) and Proposition 3.3.3.

(A3) We prove this in the proof of Proposition 3.1.3. $\square$

Let

$$\Gamma'(\theta) := I/(\tau),$$

where τ is the involution on I induced by that of $\Gamma\times\Gamma$ permuting the factors. Note that $I \to \Gamma(\theta)$ factor through $\Gamma'(\theta)$.

Corollary 3.5.2. *For* $(\mathcal{H}_1,\theta)$*, it holds* $\Gamma(\theta)' \simeq \Gamma(\theta)$*.*

Proof. By Lemma 3.5.1, (A1) holds for $(\mathcal{H}_1,\theta)$. Thus, by [DK93, Corollary 7.1.7], we have $p_a(\Gamma(\theta)') = \frac{3}{2}g(g-1)+1$. Thus, by Corollary 3.3.4, $p_a(\Gamma(\theta)') = p_a(\Gamma(\theta))$. By (A1) again, the natural morphism $\Gamma(\theta)' \to \Gamma(\theta)$ is birational. Therefore it holds $\Gamma(\theta)' \simeq \Gamma(\theta)$. □

By a moduli argument we prove the conjecture for a general pair (Γ,θ').

Theorem 3.5.3. *A general spin curve satisfies the conditions* (A1)–(A3)*. In particular, the Scorza quartic exists for a general spin curve.*

Proof. Let $\mathcal{M}$ be the moduli space of couples (Γ,θ'), where Γ is a curve of genus g and θ' is a theta characteristic such that $h^0(\Gamma,\theta')=0$. Classically, $\mathcal{M}$ is known to be irreducible (see [Cor]). Let U be a suitable finite cover of an open neighborhood of a couple $(\mathcal{H}_1,\theta)$ such that there exists the family $\mathcal{G}\to U$ of pairs of canonical curves and non-effective theta characteristics. Denote by (Γ_u,θ_u) the fiber of $\mathcal{G}\to U$ over $u\in U$. By Lemma 3.5.1, $(\mathcal{H}_1,\theta)$ satisfies (A1)–(A3). Since the conditions (A1) and (A3) are open conditions, these are true on U. Thus we have only to prove that the condition (A2) is still true on U. Let $\mathcal{J}\to U$ be the family of Jacobians and $\Theta\to U$ the corresponding family of theta divisors. By [DK93, p.279-282], the family $\mathcal{I}$ of theta-correspondences embeds into Θ, and by the family of Gauss maps $\Theta\to\check{\mathbb{P}}^{g-1}\times U$, we can construct the family $\widetilde{\mathcal{G}}\to U$ whose fiber $\widetilde{\mathcal{G}}_u\subset\check{\mathbb{P}}^{g-1}$ is the discriminant $\Gamma(\theta_u)$. By Corollary 3.5.2, it holds $\Gamma(\theta)'\simeq\Gamma(\theta)$ for $(\mathcal{H}_1,\theta)$. Thus we have also $\Gamma(\theta_u)'\simeq\Gamma(\theta_u)$ for $u\in U$. By [DK93, Corollary 7.1.7], we see that $p_a(\Gamma(\theta_u))$ and $\deg\Gamma(\theta_u)$ are constant for $u\in U$. Thus $\widetilde{\mathcal{G}}\to U$ is a flat family since the Hilbert polynomials are constant. Since no quadric contains $\Gamma(\theta)$ for $(\mathcal{H}_1,\theta)$, neither does $\Gamma(\theta_u)$ for $u\in U$ by the upper semi-continuity theorem. □

Remark. Let (Γ,θ') be a general pair of a canonical curve Γ and a non-effective theta characteristic θ'. In the proof of Theorem 3.5.3, we prove that $\Gamma(\theta')'\simeq\Gamma(\theta')$.

3.6. F_4 **is the Scorza quartic for** $(\mathcal{H}_1,\theta)$**.**

Note that, for F_4, it holds $\mathcal{D}_2 = \mathrm{CP}(F_4)_{|\mathcal{H}_2\times\mathcal{H}_2}$ and $\widetilde{D}_q = P_{H_q^2}(\check{F}_4)$.

Let $\{F_4'=0\}$ be the Scorza quartic for $(\mathcal{H}_1,\theta)$. Let D_H' be defined as in 3.4 for F_4'. We simply denote D'_{H_q} by D'_q.

Proposition 3.6.1. F_4 *is the Scorza quartic for* $(\mathcal{H}_1,\theta)$*.*

Proof. The assertion is equivalent to the following equality:

$$\mathrm{CP}(F_4')_{|\mathcal{H}_2\times\mathcal{H}_2} = \mathrm{CP}(F_4)_{|\mathcal{H}_2\times\mathcal{H}_2}.$$

Since both sides have the structures of quadric section bundles over $\mathcal{H}_2$, it suffices to prove $\{D_q'=0\}=\{\widetilde{D}_q=0\}$ for a generic q. Since there is no quadric containing $\Gamma(\theta)$ by Lemma 3.5.1, it is sufficient to show that $\{D_q'=0\}\cap\Gamma(\theta)=\{\widetilde{D}_q=0\}\cap\Gamma(\theta)$. The set $\{\widetilde{D}_q=0\}\cap\Gamma(\theta)$ consists of points corresponding to the line pairs intersecting q. On the other hand, $\{D_q'=0\}\cap\Gamma(\theta)=\overline{D}_{H_q}$ by the definition of the Scorza quartic

since H_q is reduced for a general q. By the definition of $\overline{D}_{H_q}$, we can easily show the set $\overline{D}_{H_q}$ also consists of points corresponding to the line pairs intersecting q. Thus $\{D'_q = 0\} \cap \Gamma(\theta) = \{\widetilde{D}_q = 0\} \cap \Gamma(\theta)$ as desired. □

3.7. **Moduli space of trigonal spin curves.**

As in Mukai's case we can reconstruct the threefold $\widetilde{A}$, that is the couple (B, C), via the curve $\mathcal{H}_1$ and a non-effective theta characteristic θ on it.

Proposition 3.7.1. *$\widetilde{A}$ is recovered from $(\mathcal{H}_1, \theta)$.*

Proof. From $(\mathcal{H}_1, \theta)$, we can define $\Gamma(\theta)$ as in Definition 3.3.1 and F_4 by Proposition 3.6.1. By Theorem 2.4.18 and Proposition 3.3.3, $\mathcal{H}_2$ is recovered from $\Gamma(\theta)$ as the intersection of cubics containing $\Gamma(\theta)$. By Theorem 2.5.12 and Lemma 2.5.14, $\widetilde{A}$ is recovered from F_4 and $\mathcal{H}_2$. □

For the next result, we denote $\widetilde{A}$ by $\widetilde{A}_d$.

Recall that we denote by $\mathcal{H}^B_d$ the union of components of the Hilbert scheme of B whose general points parameterize smooth rational curves of degree d obtained inductively as in Proposition 2.2.1. By the remark after the proof of Proposition 3.7.5, $\mathcal{H}^B_d$ is irreducible.

We identify $\mathcal{H}^B_d$ with the moduli space of $\widetilde{A}_d$, which we denote by $\mathcal{M}_d$. Let $\mathcal{M}'_g$ and $\widetilde{\mathcal{M}}'_g$ be the moduli space of trigonal curves of genus g and the moduli space of pairs of trigonal curves of genus g and even theta characteristics, respectively. We can define the rational map $\pi_{\mathcal{M}} \colon \mathcal{M}_d \dashrightarrow \widetilde{\mathcal{M}}'_{d-2}$ by setting $\widetilde{A}_d \mapsto (\mathcal{H}_1, \theta)$.

Corollary 3.7.2. *$\pi_{\mathcal{M}}$ is birational. Moreover, $\operatorname{Im}\pi_{\mathcal{M}}$ is an irreducible component of $\widetilde{\mathcal{M}}'_{d-2}$ dominating $\mathcal{M}'_{d-2}$. In particular a general $\mathcal{H}_1$ is a general trigonal curve of genus $d-2$.*

Proof. The first assertion follows from Proposition 3.7.1.

Since $\dim \mathcal{H}^B_d = 2d$ and $\dim \operatorname{Aut}(B, C_d) \leq 3$, we see that $\dim \mathcal{M}_d \geq 2d-3$. On the other hand, $\dim \mathcal{M}'_{d-2} = 2d-3$ and a smooth curve has only a finite number of theta-characteristics. Thus the latter part follows from the first. □

Combining Theorem 2.5.12, Proposition 3.6.1 and Corollary 3.7.2, we obtain:

Corollary 3.7.3. *Let F_4 be the Scorza quartic for a general trigonal spin curve of genus $d-2\,(d \geq 6)$ and $\mathcal{H}_2$ the intersection of cubics containing the discriminant locus of the trigonal spin curve. Set $n := \frac{(d-1)(d-2)}{2}$. Then the normalization of the main component of $\operatorname{VSP}(F_4, n; \mathcal{H}_2)$ is isomorphic to the blow-up of quintic del Pezzo threefold B along a general smooth rational curve of degree d and then the strict transforms of its bi-secant lines on B.*

By our study we see some other problems which are of a certain interest:

Problem 3.7.4. (1) Is the Hilbert scheme of curves of B whose generic point corresponds to a smooth rational curve of degree d irreducible, namely, is $\mathcal{H}^B_d$ the unique irreducible component ?

(2) Is $\widetilde{\mathcal{M}}'_g$ irreducible ?

If $d = 5$, (2) is true by [DK93, Lemma 7.7.1]. We show that (1) is true also for $d \leq 6$. (Probably if $d \leq 5$, then it is known. Our contribution is for $d = 6$).

Proposition 3.7.5. *If $d \leq 6$, then the answer to Problem 3.7.4 (1) is affirmative.*

Proof. For a smooth projective variety X in some projective space, let $\mathcal{C}_d^0(X)$ be the components of the Hilbert scheme of X whose general points parameterize smooth rational curves of degree d. By [Per02], $\mathcal{C}_d^0(G(a,b))$ is irreducible, where $G(a,b)$ is the Grassmannian parameterizing a-dimensional sub-vector spaces in a fixed b-dimensional vector space. The claim is that $\mathcal{C}_d^0(\mathbb{P}^6 \cap G(2,5))$ is irreducible, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to $G(2,5)$. The claim is true for $d = 1$ since $\mathcal{H}_1^B \simeq \mathbb{P}^2$.

Let $\mathcal{B}$ be the irreducible family of del Pezzo 3-folds $B = G(2,5) \cap \mathbb{P}^6$, where $\mathbb{P}^6 \subset \mathbb{P}^9$ is transversal to $G(2,5)$. Let

$$J = \{([C_d^0],[B]) \in \mathcal{C}_d^0(G(2,5)) \times \mathcal{B} \mid C_d^0 \subset B\}.$$

The claim is equivalent to show that a general fiber $J \to \mathcal{B}$ is irreducible. Since $d \leq 6$, a smooth rational curve of degree d is contained in at least a six-dimensional projective space. Thus a general fiber of $J \to \mathcal{C}_d^0(G(2,5))$ is non-empty and irreducible. Since $\mathcal{C}_d^0(G(2,5))$ is irreducible, it holds J is irreducible. By the argument of [MT01, Proof of Theorem 3.1 p.17], we have only to show that there is one particular component $\mathcal{C}_d^{0\star}(B)$ of a general fiber $J \to \mathcal{B}$ invariant under monodromy.

By induction let us assume that $\mathcal{C}_{d-1}^0(B)$ is irreducible. Let $[C_{d-1}^0] \in \mathcal{C}_{d-1}^0(B)$ be a generic element. The family of lines $[l] \in \mathcal{H}_1^B$ which intersect a general element of $\mathcal{C}_{d-1}^0(B)$ is irreducible by Proposition 2.2.4 (3). This implies that the family $\mathcal{C}_{d-1,1}^0(B)$ of reducible curves $C_d^0 = C_{d-1}^0 \cup l$ such that $[C_{d-1}^0] \in \mathcal{C}_{d-1}^0(B)$, $[l] \in \mathcal{H}_1^B$ and length $C_{d-1}^0 \cap l = 1$ is irreducible. Similarly to the proof of Proposition 2.2.1, we see that the locus containing the points corresponding to the smoothings of curves from $\mathcal{C}_{d-1,1}^0(B)$ is an irreducible component of J. □

Remark. The proof of the proposition shows that $\mathcal{H}_d^B$ is irreducible for any d.

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